Multiparticle production and statistical analogies

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A study of the singularities and zeros of the generating functions of the multiplicity distributions is advocated. Some hints from well-known probability distributions and experimental data are given. The statistical mechanics analogies call for a look for a signature of the phase transitions. A program for further experimental studies of the singularities is formulated. © *1994 American Institute of Physics.*

Multiplicity distributions in high-energy collisions of various projectiles and targets have a qualitatively similar (but quantitatively different!) behavior. That is why many fits by well-known probability distributions have been tried. The progressively more sensitive characteristics, such as the ratio of the cumulant to factorial moments, which reveal new features in the experimental data,[2] have been proposed.[1] Their understanding requires further experimental and theoretical studies. We suggest that more attention should be focused on the structure of singularities and zeros of the generating functions of the multiplicity distributions, which is especially appealing because of the possible statistical analogies.[3–6]

We define the generating function $G(z)$ of the probability distribution P_n by the relation

$$G(z)=\sum_{n=0}^{\infty}(1+z)^n P_n. \tag{1}$$

In what follows, we often use the function

$$\Phi(z)=\ln G(z). \tag{2}$$

The (normalized) factorial (F_q) and the cumulant (K_q) moments of the distribution P_n are related to them by the formulas

$$G(z)=\sum_{q=0}^{\infty}\frac{z^q}{q!}\langle n\rangle^q F_q \quad (F_0=F_1=1), \tag{3}$$

$$\Phi(z)=\sum_{q=1}^{\infty}\frac{z^q}{q!}\langle n\rangle^q K_q \quad (K_1=1), \tag{4}$$

where $\langle n\rangle$ is the average multiplicity.

First, we consider certain distributions which give analytical examples of the singularities. We start with the fixed multiplicity (FM) distribution, when the sample of events of the same multiplicity (n_0) is chosen. We then proceed to the Poisson distribution (P) as a reference to independent emission processes and, finally, consider the gamma (Γ), the negative binomial (NB), and the lognormal (L) distributions which are widely used to fit the experimental data at high energies. The corresponding functions $\Phi(z)$ are

$$\Phi^{FM}(z)=n_0\ln(1+z), \tag{5}$$

$$\Phi^{P}(z)=\langle n\rangle, \tag{6}$$

$$\Phi^{\Gamma}(z)=-\mu\ln\left[1-\frac{\langle n\rangle}{\mu}\ln(1+z)\right], \tag{7}$$

$$\Phi^{NB}(z)=-k\ln\left(1-\frac{z\langle n\rangle}{k}\right), \tag{8}$$

where μ and k are the adjustable parameters. The lognormal distribution is the only one which is not determined by its moments. From the integral representation of its generating function,

$$\Phi^{L}(z)\rightarrow-\ln\int_0^{\infty}\exp\left[-\frac{(\ln x-\nu)^2}{2\sigma^2}+x\ln(1+z)\right]d(\ln x), \tag{9}$$

we easily see that its convergence radius is given by the inequality

$$|z+1|_L\leqslant 1; \tag{10}$$

i.e., the singularities come close to the point $z=0$, but they are "soft" in the sense that the normalization condition $G(0)=1$ persists. For other distributions the nontrivial (essential for our purposes) singularities are situated at

$$z_{NB}=k/\langle n\rangle, \tag{11}$$

$$z_{\Gamma}=\exp(\mu/\langle n\rangle)-1, \tag{12}$$

$$z_P=\infty, \tag{13}$$

$$z_{FM}=-1. \tag{14}$$

We note that NB and Γ singularities are close to $z=0$ if the parameters k and μ are much less than $\langle n\rangle$. It is especially interesting because the factorial and the cumulant moments are calculated as the qth derivatives of $G(z)$ and $\Phi(z)$ at that point and the nearby singularity influences their behavior substantially. In particular, it is important that the ratio of the moments

$$H_q=K_q/F_q, \tag{15}$$

which is identically equal to zero for the Poisson distribution, alternate sign at each rank in the case of fixed multiplicity, and that it always be positive for Γ and NB, which tend at asymptotically large ranks to zero as,[7] q^{-k}. Different types of behavior are predicted in QCD with a strong decrease at low ranks, followed by (quasi)oscillations at larger ranks.[1,7,8] It would be interesting to determine what singularity governs such a shape. This problem has not yet been solved.

Let us set up some guidelines for the experimental data. In experiments with different projectiles and targets the adjustable parameters are different and energy dependent. We can nevertheless obtain a qualitative estimate of the approximate locations of the singular points. In the e^+e^- collisions, the *NB* estimates give rise to $k/\langle n\rangle \sim 1$ (see, e.g., Ref. 9) and, therefore, the singularity is situated at $z_{ee} \sim 1$, i.e., rather far from $z=0$. It is much closer to the origin in the *hh* collisions, where (see, e.g., Ref. 10) $k/\langle n\rangle \sim 10^{-1}$. The *AA* data are not so definite[11] (even though the lower statistics is slightly compensated for by the larger multiplicity) and give rise to $k/\langle n\rangle \leq 10^{-1}$ and, thus, to ever closer (to the origin) singularity. The singularities move to the origin with an increase in energy. These qualitative tendencies are probably related to similar regularities in the behavior of the depth of the minimum of H_q found for various reactions (see Refs. 2 and 11) and to oscillations of H_q at large q (see below). The oscillations of experimental distributions about the smooth *NB* fit (see, e.g., Ref. 9) also could be connected with those oscillations. Their physical meaning could correspond to various objects (ladders, etc.) which contribute to different multiplicities and which should be checked in the Monte Carlo models. Another possible source of oscillations due to the cutoff of the multiplicity tail by the conservation laws should die out asymptotically.[12]

However, this cutoff plays an important role when one tries to restore the generating function directly from experimental data. Actually, the series (1) is replaced now by the partial sum in the form of the polynomial in z:

$$G_N(z) = \sum_{n=0}^{N} (1+z)^n P_n \tag{16}$$

with N equal to the highest observed multiplicity. Therefore, the truncated generating function $G_N(z)$ has N complex conjugate zeros,

$$G_N(z) = \prod_{j=1}^{N} \left(1 - \frac{z}{z_j}\right). \tag{17}$$

It was shown by DeWolf[13] that the zeros cover a circle in the complex z plane for the *ee* events generated by JETSET Monte Carlo program at 1000 GeV. It reminds us of the Lee–Yang zeros[3] in statistical mechanics. They seem to close in onto the singularity of $G(z)$ at some real $z=z_s>0$ as N increases.

It is known[14] that the degree of infinity k of $G_N(z_s)$ is the same as the order of singularity of $G(z)$ at $z=z_s$ in the case of algebraic-logarithmic behavior and (for the algebraic singularity) is determined by the slope on the double-log plot in the limit $N \to \infty$:

$$\ln G_N(z_s) \to k \ln N + \ln[A_k/\Gamma(k+1)]. \tag{18}$$

Here A_k is the residue of $G(z)$ at $z=z_s$ (NBD gives an example; see Ref. 8).

Also, the order ρ of the integer function is given[15] by the formula

$$\lim_{n\to\infty} \frac{-\ln P_n(1+z_s)^n}{n \ln n} = \frac{1}{\rho}. \tag{19}$$

The formulas given above can be used to interpret experimental data.

The cumulants are determined[5,13] by the moments of zero locations,

$$K_q = -\frac{(q-1)!}{\langle n\rangle^q} \sum_{j=1}^{N} z_j^{-q} = -\frac{(q-1)!}{\langle n\rangle^q} \sum_{j=1}^{N} \frac{\cos q\theta_j}{r_j^q}, \tag{20}$$

where we denote $z_j = r_j \exp(i\theta_j)$. Thus, the oscillations mentioned above are related to the phases of zeros.

The study of singularities of the generating function becomes more beneficial if one uses statistical mechanics analogies and recalls the Feynman fluid model.[4–6] The generating function is analogous to the partition function of the grand canonical ensemble and $\Phi(z)$ is analogous to the free energy. The total rapidity range plays the role of the volume and the variable $1+z$ is the fugacity. The "pressure" $p(z)$ and the mean number of particles at a given fugacity $\langle n(z)\rangle$ (which is proportional to the usual pressure and density) can be defined by the formulas

$$p(z) = \lim_{Y\to\infty} \frac{\Phi_N}{Y}, \tag{21}$$

$$\langle n(z)\rangle_N = (1+z)\frac{\partial\Phi_N}{\partial z}, \tag{22}$$

where $\Phi_N(z) = \ln G_N(z)$ and $\langle n(0)\rangle_N = \langle n\rangle$. We note that the behavior of $\langle n(z)\rangle_N$ in the complex z plane, which was determined from the experimental data, should easily reveal the zeros z_j of the function G_N (Ref. 5), since it has poles exactly at the same loci z_j

$$\langle n(z)\rangle_N = \sum_{j=1}^{N} \frac{1+z}{z-z_j}. \tag{23}$$

The plots of $p(z)$ from experimental data for *ee* and *hh* reactions extrapolated to $Y\to\infty$ were shown in Ref. 6. We found that the latest LEP data (e.g., Ref. 9) coincide well with the extrapolation used in Ref. 6 before the LEP data became available. The authors of Ref. 6 claim that there is no phase transition in the *ee* collisions. A qualitative conclusion from Figs. 3a and 3b of Ref. 6 is that $p(z)$ increases at $z>0$ much faster in the non-diffractive *hh* collisions as compared to the *ee* collisions. It demonstrates that the *hh* singularity is closer to the origin that corresponds to the conclusions reached above. The increase would be even more drastic in the case of *AA* collisions [the data of EMU01 (Ref. 11) were used for estimates], but it is strongly influenced by single events with very high multiplicity. Thus the *AA* analysis is hard to extend to large z. Probably, it has a physical origin, since the *AA* collisions are most likely responsible for the phase transitions. The constancy of $p(z)$ at $z<0$ for *hh* collisions in Fig. 3b of Ref. 6 is quite unexpected. In statistical mechanics it would be a signature for a phase transition. If supported by further studies, it would justify theoretical speculations. The problem of phase transition in systems with relatively small number of particles should be treated carefully. In particular, it depends on the steepness of the increase of $p(z)$ with z. Some of its criteria are yet to be published. However, the similarities may well turn out to be mainly of formal nature and just the methods of analysis may be comparable. Nevertheless, some physical models based on the analogy have been proposed.[16–19]

Our preliminary qualitative results allow us to formulate a further program of analysis of experimental data, which consists of determining:

1) the radius of convergence of G_N (1) according to Cauchy ($P_n^{1/n}$) and D'Alembert (P_n/P_{n-1}) criteria;

2) the approach to the Carleman condition $\Sigma_{n=1}^{\infty} F_{2n}^{-1/2n} = \infty$ at high energies ($N \to \infty$);

3) location of zeros of $G_N(z)$ [Eqs. (17) or (23)] and their density;

4) the order of the singularity of $G(z)$ and its residue, (18);

5) the order of the integer function, (19);

6) the behavior of the "pressure" $p(z)$, (21);

7) the behavior of the "multiplicity" $\langle n(z) \rangle$, (22);

8) the higher derivatives of Φ_N (the fractional derivatives can also be used,[20] especially in connection with the classification of the phase transitions of noninteger order proposed recently[21]).

The extrapolations to $Y \to \infty$ should be attempted. It is possible that zero locations will differ for different classes of processes (diffractive and nondiffractive; two and three jets, etc.). The drastic change in the behavior of Φ_N or its derivatives must be carefully analyzed to determine a possible presence of a signature of the phase transition. Its theoretical criteria in finite systems should be developed in parallel. We hope that the first stage of the program formulated above can provide some new insights into the physics of multiparticle production. More detailed results will be published elsewhere.

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