Statistical properties of eigenfunctions in a disordered metallic sample

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The distribution of eigenfunction amplitudes and the variance of the "inverse participation ratio" (IPR) in disordered metallic samples have been calculated. The weak-localization corrections to the predictions of the random matrix theory are found. © *1994 American Institute of Physics.*

The statistical properties of disordered metals have attracted a considerable research interest in recent years. It was understood that the old problem of a quantum particle moving in a quenched random potential, considered earlier in the context of the Anderson localization and mesoscopic phenomena,[1] exemplifies a particular class of chaotic quantum systems and has much in common with such paradigmatic problems in the domain of quantum chaos as quantum billiards.[2] The Wigner–Dyson energy level statistics, first found in the framework of random matrix theory (RMT)[3] and considered to be a "fingerprint" of quantum chaotic systems,[4] was also shown to be relevant for disordered metals.[5,6] This fact gave rise to a broad application of RMT results for qualitative and quantitative descriptions of mesoscopic conductors and stimulated a common interest in statistical characteristics of spectra of disordered systems.[7]

At the same time, less attention was given to statistical properties of eigenfunctions in disordered or chaotic quantum systems. Recently, however, the distribution of eigenfunction amplitudes was shown to be relevant for description of fluctuations of tunneling conductance across the "quantum dots,"[8] as well as for the understanding of some properties of atomic spectra.[9] In addition, a so-called "microwave-cavity" technique has emerged[10] as a laboratory tool to simulate a disordered quantum system. This technique, which allows us to observe directly the eigenfunction spatial fluctuations, was used in Ref. 11 to study experimentally the eigenfunction statistics in weak-localization regime. All these facts make us focus special attention on the eigenfunction statistics, which must be studied in detail theoretically.

In order to characterize eigenfunction statistics quantitatively, it is convenient to introduce a set of moments $I_q=\int|\psi(r)|^{2q}d^dr$ of eigenfunction local intensity[12] $|\psi(r)|^2$. The second moment I_2 is known as the inverse participation ratio (IPR). This quantity is a useful measure of the eigenfunction localization: it is inversely proportional to the volume of a part of the system which contributes effectively to the eigenstate normalization. For completely "ergodic" eigenfunctions which cover randomly, but uniformly, the whole sample $I_2 \propto 1/V$, where V is the volume of the system. If, in contrast, the eigenfunctions are localized, i.e., concentrated in the region of linear size ξ, the mean IPR scales are $\bar{I}_2 \propto \xi^{-D_2}$, where D_2 is an effective dimension which can be different from a spatial dimensionality d because of the multifractal structure of eigenfunctions.[12] Correspondingly, the IPR fluctuations reflect *level-to-level* variations of the spatial structure of the eigenfunction.

The most complete analytical study of statistical characteristics of eigenfunctions was performed for the cases of $0d$ systems[13,14] and for strictly 1D (Ref. 15) and quasi-1D (Refs. 16 and 17) geometry. Some analytical results were obtained also for a system in the vicinity of the localization transition in the dimensionality $d=2+\epsilon$, $\epsilon \ll 1$ (Ref. 12) and for $d \to \infty$ (Ref. 18). Let us note that in Refs. 13, 14, and 16–18 the supersymmetry method was used. This is a very powerful tool for studying distribution functions of various quantities which characterize the eigenfunction statistics.

In the present letter we address systematically the issue of the eigenfunction statistics for arbitrary spatial dimensionality d in the weak localization domain. In the leading approximation (which ignores spatial structure of the system and treats it as a zero-dimensional system) these statistics are described by the RMT which predicts a Gaussian distribution of the eigenfunction amplitudes[13,17] $\psi(r)$. We know since the publication of a paper by Altshuler and Shklovskii[6] that the diffusion motion of a particle in a metallic sample produces deviations of spectral statistics from what can be expected in RMT. To the best of our knowledge, the analogous problem for the eigenfunction statistics in 2D and 3D systems has never been studied. It is considered only in this paper. We use a recently developed method,[19] which is based on the supersymmetry methods[5,20] and which combines a perturbative elimination of fast diffusive modes (in the spirit of the renormalization group ideas) and a consequent nonperturbative evaluation of the resulting $0d$ integral. We can thus calculate the deviations from the Gaussian distribution of $\psi(r)$ in mesoscopic metallic samples. We can also calculate the variance of the IPR, which turns out to be on the order of $1/g^2$, where g is the dimensionless (measured in units of e^2/h) conductance of the sample.

In order to calculate the distribution of the eigenfunction amplitude and to find the IPR variance, we use the fact that relevant quantities can be expressed in terms of the correlation functions of a certain supermatrix σ-model.[5,20] A quite general exposition of the method, which is not repeated here, can be found in Ref. 21. Depending on whether the time reversal and spin rotation symmetries are broken, one of three different σ-models is relevant, with the orthogonal, unitary, or symplectic symmetry group. We consider mostly the case of the unitary symmetry throughout this paper. For two other cases the calculations are similar, and only the results are presented.

The expressions for $\bar{I}_q$ and $\overline{I_2^2}$ (the bar stands for disorder averaging) in terms of the σ-model read as follows:

$$\bar{I}_q = \frac{-1}{2V} \lim_{\epsilon \to 0} \epsilon^{q-1} \left.\frac{\partial}{\partial u}\right|_{u=0} \int DQ \exp\left\{ -\mathcal{F}^{(q)}(u,Q) + \frac{1}{t} \int d^d r \operatorname{Str}(\nabla Q)^2 \right\}, \quad (1)$$

$$\overline{I_2^2}=\frac{-1}{6V}\lim_{\epsilon\to 0}\epsilon^3\left.\frac{\partial^2}{\partial u^2}\right|_{u=0}\int DQ\exp\left\{-\mathcal{F}^{(2)}(u,Q)+\frac{1}{t}\int d^dr\,\mathrm{Str}(\nabla Q)^2\right\},\tag{2}$$

where

$$\mathcal{F}^{(q)}(u,Q)=\int d^dr\{\epsilon\,\mathrm{Str}(\Lambda Q)+u\,\mathrm{Str}^q(Q\Lambda k)\},$$

$$Q=T^{-1}\Lambda T,\quad \Lambda=\mathrm{diag}(1,\ 1,\ -1,\ -1),$$

$$k=\mathrm{diag}(1,\ -1,\ 1,\ -1),\quad \frac{1}{t}=\pi\nu\mathcal{D}/4.\tag{3}$$

Here T is a 4×4 supermatrix which belongs to the coset space $U(1,1|2)$, $\mathcal{D}$ is the classical diffusion constant, ν is the density of states, and V is the volume of the system.

In general, the RMT predictions are applicable to a disordered metallic system under the following conditions: $L\gg l$; $E_c\gg\Delta$, where L is the size of the system, l is the mean free path, $E_c=\hbar\mathcal{D}/L^2$ is the Thouless energy, and Δ is the mean level spacing. The eigenfunctions for such systems are known to be ergodic, where the amplitudes $\psi(r)$ are uncorrelated (for $|r-r'|\gtrsim l$) Gaussian-distributed complex (real) variables for a broken (unbroken) time-reversal symmetry, respectively. This immediately gives[17,14] $\overline{I_q^{(u)}}=q!/V^{q-1}$ and $\overline{I_q^2}=\overline{I_q}^2$, where the superscript u refers to the unitary symmetry.

In the framework of the σ-model formalism these results can be easily reproduced if any spatial variation of the supermatrix field $Q(r)$ is ignored. Equations (1) and (2) will then reduce to integrals over a single supermatrix which can be evaluated exactly. The corrections to the RMT results have the form of a regular expansion in a small parameter $\Delta/E_c=g^{-1}$. A systematic method of constructing such an expansion can be briefly outlined as follows.[19] The matrix $Q(r)$ is decomposed as $Q(r)\ =\ T_0{}^{-}{}^{1}\ -\ 1\ -\ 1\ -\ 1\ -\ 1\ -\ 1\ -\ 1\tilde{Q}(r)T_0$, where T_0 is a spatially uniform matrix, and $\tilde{Q}$ describes all modes with nonzero momenta. When $\Delta\ll E_c$, the matrix $\tilde{Q}$ fluctuates only weakly around the value $\tilde{Q}=\Lambda$. Thus, it can be expanded as $\tilde{Q}=\Lambda(1+W+w^2/2+\dots)$, where W is a block, off-diagonal supermatrix which represents independent fluctuating degrees of freedom. Substituting this expansion into Eqs. (1)–(3) and integrating out the "fast" modes, we obtain an expression for the renormalized functional $\mathcal{F}_{\mathrm{eff}}^{(q)}(u,Q_0)$, where $Q_0=T_0^{-1}\Lambda T_0$ is an r-independent matrix (zero mode). The contribution of the eliminated "fast" modes is expressed in terms of the diffusion propagator $P(r_1,r_2)$. For an isolated sample this propagator has the form

$$P(r_1,r_2)=\sum_q\cos(qr_1)\cos(qr_2)P(q),$$

$$P(q)=\frac{1}{2\pi\nu V}\frac{1}{\mathcal{D}q^2+\epsilon},\quad q=\pi\left(\frac{n_1}{L_1},\ \dots,\ \frac{n_d}{L_d}\right),\tag{4}$$

$$n_i=0,\pm 1,\pm 2,\dots,\quad \sum n_i^2>0,$$

where the system is thought to be of the size $L_1 \times L_2 \times \ldots \times L_d$. Finally, the integrals over Q_0 are performed exactly.

Applying this method to Eqs. (1) and (3), we obtain

$$\overline{I_q^{(u)}} = \frac{q!}{V^{q-1}} \left\{ 1 + \frac{a_1}{g} q(q-1) + O\left(\frac{1}{g^2}\right) \right\}, \tag{5}$$

where $g = 2\pi\nu \mathcal{D} L^{d-2}$ is the conductance of the sample. The value of the coefficient $a_1 = g\Sigma_q P(q)$ depends on the spatial dimension. This value is $a_1 = 1/6$ in quasi-1D systems. For $d \geq 2$ the corresponding sum over the momenta q diverges at large $|q|$ and is the cutoff at $|q| \sim l^{-1}$. This gives $a_1 = (1/2\pi) \ln L/l$ for $d=2$ and $a_1 \propto L/l$ for $d=3$.

Knowing all the moments $\overline{I_q^{(u)}}$, it is an easy task to restore the whole probability distribution $\mathcal{P}(y)$ of the eigenfunction local intensity $y = V|\psi(r)|^2$:

$$\mathcal{P}^{(u)}(y) = e^{-y} \left[1 + \frac{a_1}{g}(2 - 4y + y^2) + O\left(\frac{1}{g^2}\right) \right]. \tag{6}$$

The corresponding equations for systems with unbroken time reversal symmetry (orthogonal and symplectic σ-model) are as follows:[22]

$$\mathcal{P}^{(0)}(y) = \frac{e^{-y/2}}{\sqrt{2\pi y}} \left[1 + \frac{a_1}{g}\left(\frac{3}{2} - 3y + \frac{y^2}{2}\right) + O\left(\frac{1}{g^2}\right) \right], \tag{7}$$

$$\mathcal{P}^{(sp)}(y) = 4ye^{-2y} \left[1 + \frac{a_1}{g}(3 - 6y + 2y^2) + O\left(\frac{1}{g^2}\right) \right]. \tag{8}$$

The leading terms here reproduce the well-known Porter–Thomas distribution, which is the RMT result.[3] The rest is the weak-localization correction. In the quasi-1D sample these expressions coincide with that obtained in Ref. 17 if the scaling parameter introduced in Ref. 17 is identified as $x = g^{-1}$. Equations (6), (7), and (8) are valid up to $y \lesssim \sqrt{g/a_1}$. For larger values of y (i.e., in the far "tail") the distribution function $\mathcal{P}(t)$ differs strongly from that of the random matrix theory and cannot be found by the method used here.

The distribution of the eigenfunction amplitude, $\mathcal{P}(t)$, was recently studied experimentally in a microwave cavity with a disorder.[11] The reported results are in good agreement with Eq. (7) which was obtained by us.

Let us now consider the IPR fluctuations. It turns out that IPR variance is on the order of $1/g^2$. Expression (5) is therefore insufficient for our needs and should be extended to the next order. Using the same method, we find

$$\overline{I_2^{(u)}} = \frac{2}{V} \left[1 + \frac{2a_1}{g} + \frac{1}{g^2}(2a_1^2 - 5a_2) + O\left(\frac{1}{g^3}\right) \right],$$

$$\overline{[I_2^{(u)}]^2} = \left(\frac{2}{V}\right)^2 \left[1 + \frac{4a_1}{g} + \frac{8a_1^2}{g^2} - \frac{2a_2}{g^2} + O\left(\frac{1}{g^3}\right) \right].$$

Here the coefficient a_2 is defined as $a_2 = g^2 \Sigma_q P^2(q)$. This coefficient is

$$a_2=\frac{1}{\pi^4}\sum_{n_s\geq 0;\; n^2>0}\frac{1}{(n_1^2+\cdots+n_d^2)^2}, \tag{9}$$

where the sum converges for $d<4$. For quasi-1D samples we have $a_2=1/90$, and the expressions which we found coincide with the results of Refs. 16 and 17.

Thus, we find the following expression for the relative variance of the IPR distribution:

$$\delta^{(u)}(I_2)\equiv\frac{\overline{[I_2^{(u)}]^2}-[\overline{I_2^{(u)}}]^2}{[\overline{I_2^{(u)}}]^2}=\frac{8a_2}{g^2}+O\left(\frac{1}{g^3}\right). \tag{10}$$

This result demonstrates that for a metallic sample the distribution function of IPR $\mathcal{P}(I_2)$ has the form of a narrow peak with a typical width on the order of $\delta^{1/2}\propto g^{-1}\ll 1$. As $g\to\infty$, $\mathcal{P}(I_2)\to\delta(I_2-\bar{I}_2)$, a result which we expect from RMT. For the orthogonal and symplectic symmetry cases we find $\delta^{(o)}(I_2)=32a_2/g^2$ and $\delta^{(sp)}(I_2)=2a_2/g^2$, respectively.

In summary, we have studied deviations of the eigenfunction statistical characteristics in a disordered metallic sample from those predicted in the random matrix theory.

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[22] For all symmetry cases, the conductance g is defined as $g=2\pi\nu\mathcal{D}/L^{d-2}$, ignoring the factor 2 due to the spin.

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