Two-dimensional solitons in discrete systems

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The structure and dynamics of 2D discrete solitons are analyzed on the basis of a 2D discrete nonlinear Schrödinger equation. The discrete nature of the situation modifies the dynamics of the corresponding 2D continuum model. A quasicollapse mechanism for energy condensation into large-amplitude discrete states is discussed. © *1994 American Institute of Physics.*

In this letter we are interested in the distinctive features in the behavior of nonlinear systems which stem from the fact that they are discrete. A discrete nature can substantially change the structure and influence the stability of localized states[1–4] and topological excitations.[5] It can have nontrivial effects on a modulational instability,[3,6] wave-collapse phenomena,[7,8] and other properties of continuum models. The dynamics of discrete systems is richer than that of the corresponding continuum models, because the latter describe only limiting cases of discrete problems. We should stress that the subjects of this letter are not conversions of continuum systems to a discrete form but *problems which are fundamentally discrete*. In addition to being of fundamental physical interest, discrete models are of interest for practical applications, such as systems of coupled optical waveguides,[3,9–11] models for energy transport in biophysical systems proposed by Davydov and Holstein, discrete models of Sheibe aggregations,[12] electrical arrays,[13,14] and systems which model the dynamics of DNA.[9,15–17] Even in the 1D case, discrete models exhibit a fairly complicated behavior. Localized 1D states in discrete nonlinear systems have recently been the subject of active research (see, for example, Refs. 1–4, 8–10, and 15–20). There has been less study of multidimensional models.[13,21–23] Our purpose in the present letter is to analyze localized 2D structures on the basis of a discrete nonlinear Schrödinger equation. This equation and other equations of similar structure describe 2D systems of coupled optical fibers, the dynamics of Sheibe aggregations, and the dynamics of envelope solitons in nonlinear lattices. In the present letter, however, we examine this equation not from the standpoint of specific applications but as an example of a discrete nonlinear system which can reveal general aspects of the dynamics of localized 2D structures. We demonstrate a new mechanism for the onset of narrow, large-amplitude states in multidimensional discrete models. We also show how the discrete nature of this system affects the stability of solitons and the phenomenon of collapse.

The basic equation of the model can be written

$$i\frac{\partial\Psi_{n,m}}{\partial t}+\Psi_{n+1,m}+\Psi_{n-1,m}+\Psi_{n,m+1}+\Psi_{n,m-1}-4\Psi_{n,m}+2|\Psi_{n,m}|^2\Psi_{n,m}=0. \quad (1)$$

Equation (1) has the Hamiltonian structure

$$i\frac{\partial\Psi_{n,m}}{\partial t}=\frac{\partial H}{\delta\Psi^*_{n,m}} \quad (2)$$

with the Hamiltonian

$$H=\sum_{n,m}|\Psi_{n,m}-\Psi_{n-1,m}|^2+\sum_{n,m}|\Psi_{n,m}-\Psi_{n,m-1}|^2-\sum_{n,m}|\Psi_{n,m}|^4=\text{const.} \quad (3)$$

In addition, Eq. (1) conserves the quantity $P=\Sigma|\Psi_{n,m}|^2$. To show that the properties of a discrete nonlinear system may depend on the dimensionality of the problem, we first consider the continuum limit of Eq. (1). This limit can be found in the case of "broad" distributions, which involve many modes. Introducing the coordinate x in the "n" direction, and y in the "m" direction, we find a continuum approximation for Eq. (1):

$$iU_t+U_{xx}+U_{yy}+2|U|^2U=0. \quad (4)$$

This is the well-known 2D Schrödinger equation, which describes (in particular) the steady-state self-focusing of light beams. Let us briefly review the basic properties of Eq. (4). The integrals of motion mentioned above have continuum analogs,

$$P=\int|U|^2dxdy \quad \text{and} \quad H=\int(|U_x|^2+|U_t|^2)dxdy-\int|U|^4dxdy=I_1-I_2.$$

A so-called virial theorem holds for Eq. (4) (Ref. 24):

$$\partial_t^2\int(x^2+y^2)|U|^2dxdy=8H. \quad (5)$$

Since H is a conserved quantity, this equation can be solved:

$$\langle R^2\rangle=\int(x^2+y^2)|U|^2dxdy=4Ht^2+At+B,$$

where

$$A=\frac{d\langle R^2\rangle}{dt}\Big|_{t=0} \quad \text{and} \quad B=\langle R^2\rangle|_{t=0}.$$

It is easy to show that, if the integral H is negative on some initial distribution, then $U(t,x,y)$ becomes singular over a finite time. The condition for collapse can be reformulated as a condition on the quantity P (the beam power in the optical case):

$$H=I_1-I_2\geqslant I_1-\frac{P}{P_{\text{cr}}}I_1=I_1\left(1-\frac{P}{P_{\text{cr}}}\right). \quad (6)$$

Here we are using the known inequality

$$\int |U|^4 dxdy \leqslant \frac{P}{P_{\rm cr}} \int (|U_x|^2+|U_y|^2)dxdy, \tag{7}$$

where $P_{\rm cr}$ is the value of the integral P for the basic soliton solution of Eq. (4). It is thus clear that in the case $P<P_{\rm cr}$, and without any special "fine focusing" of the original wave packet, collapse does not occur. In the case $P>P_{\rm cr}$, H can be negative, and one can show that a singularity of the wave field forms over a finite time. In the course of the self-focusing, a critical power $P=P_{\rm cr}$ is asymptotically reached. We need to stress that a singularity cannot form in the discrete case, because the integral $P=\Sigma_{n,m}|\Psi_{n,m}|^2$ is conserved. As a result of compression in the initial stage, all the energy may condense in a few modes. A self-localization of energy which was initially distributed over a nonlinear discrete system was studied in Refs. 7, 8, 11, and 15. This problem has attracted attention because of (in particular) the important role which localized, narrow, large-amplitude states may play in the dynamics of DNA[16,17] and in nonlinear optics.[11] In discrete 1D systems, the following would appear to be a typical scenario for the onset of narrow states: In the first step, initial perturbations cluster as the result of the onset of a modulational instability. Small-amplitude solitons appear in this step. In the next step, inelastic collisions of solitons lead to a transfer of energy from relatively small-amplitude solitons to relatively large-amplitude ones. In the final step, large-amplitude self-localized states form.[15] We should point out that wave collapse is a typical version of wave dynamics in multidimensional nonlinear systems, in contrast with the 1D case. The onset of a singularity in multidimensional continuum models corresponds to the condensation of all the energy in a few modes in discrete nonlinear systems. This mechanism for energy localization is quite common in multidimensional systems and is relatively insensitive to the details of the particular model. Quasicollapse thus plays the role of the mechanism for the formation of very narrow self-localized states[7,8,25] from initially broad wave packets for many nonlinear discrete systems.

Let us examine some steady-state solutions of our basic model, (1). We write them as $\Psi_{n,m}=F_{n,m}\exp(i\lambda^2 t)$, where the envelope $F_{n,m}$ is given by

$$F_{n+1,m}+F_{n-1,m}+F_{n,m+1}+F_{n,m+1}-(4+\lambda^2)F_{n,m}+2|F_{n,m}|^2F_{n,m}=0. \tag{8}$$

This equation can be thought of as a nonlinear algebraic eigenvalue problem for λ^2 and $F_{n,m}$. Soliton solutions arise from a balance between nonlinear and dispersive effects, modified because of the discrete nature of this system. In the present letter we focus on an analysis of the case in which there is a finite number of coupled equations with homogeneous or periodic boundary conditions: $\Psi_{-N,m}=\Psi_{N,m}$ and $\Psi_{n,-M}=\Psi_{n,M}$. To prove that solutions of Eq. (8) exist it is sufficient to consider the problem of minimizing H at a fixed P (see also Refs. 8 and 9). This approach has the advantage that at the same time we prove the stability of the solutions which are found with respect to perturbations that conserve P. The boundedness of Hamiltonian H follows from the chain of inequalities

$$H=\sum_{n,m}|\Psi_{n,m}-\Psi_{n-1,m}|^2+\sum_{n,m}|\Psi_{n,m}-\Psi_{n,m-1}|^2-\sum_{n,m}|\Psi_{n,m}|^4$$

$$\geqslant -\max|\Psi_{n,m}|^2 P \geqslant -P^2. \tag{9}$$

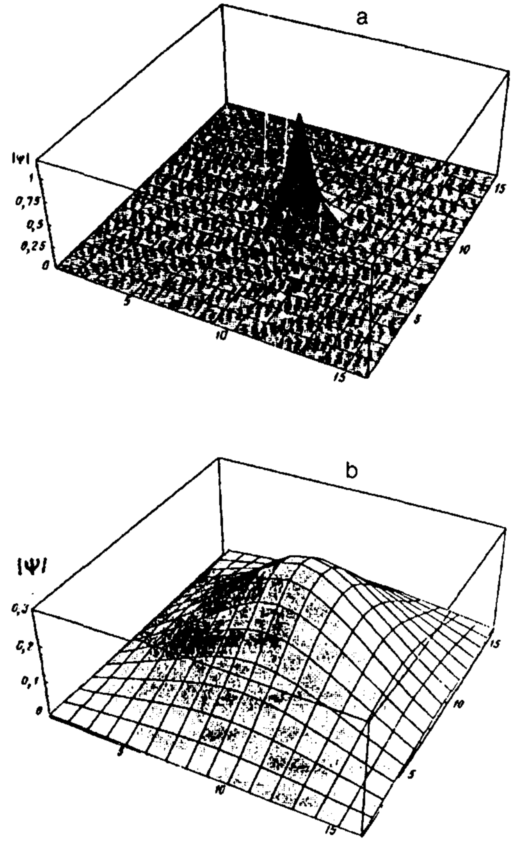

FIG. 1.

For finite values of N and M, the minimum of the Hamiltonian is reached on some solution which is obviously stable with respect to perturbations that conserve P. As was mentioned in Ref. 8, this circumstance does not mean that a solution exists for any arbitrary (continuous) λ^2, since λ^2 is a Lagrange multiplier to be determined. This fact follows from the absence of gauge invariance, which holds in the continuum limit.

Perturbation theory can be used to find one type of discrete solitons, namely, a very narrow symmetric state. We assume that nearly all the energy is localized symmetrically in only a few modes, and that the inequalities $|F_{0,0}| \gg |F_{\pm 1,0}|$, $|F_{0,\pm 1}|$ hold. The central mode is designated $F_0 = F_{0,0}$ here. By virtue of the symmetry of the problem we can write $F_{\pm 1,0} = F_{0,\pm 1} = F_1$. For the central mode we then have the equation

$$4F_1 - (\lambda^2 + 4)F_0 + 2|F_0|^2 F_0 = 0, \tag{10}$$

while for the nearest neighbors we have

$$F_0 - (\lambda^2 + 4)F_1 + 2|F_1|^2 F_1 + 3F_2 = 0. \tag{11}$$

Here F_2 represents small corrections for the effects of more-remote neighbors. In the limit of large values of λ^2 in which we are interested here, these equations lead to

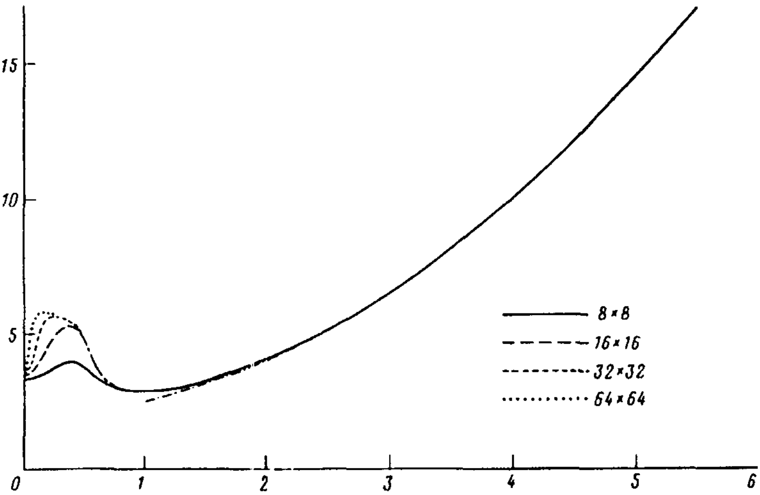

FIG. 2. The integral P as a function of λ. The dot-dashed curve corresponds to the parabola $P=\lambda^2/2+2$.

$$F_0 \approx \frac{\lambda}{\sqrt{2}} + \frac{\sqrt{2}}{\lambda} - \frac{2\sqrt{2}}{\lambda^3} + \ldots, \quad F_1 \approx \frac{1}{\sqrt{2}\lambda} + \ldots . \tag{12}$$

In this limit we have the integral $P \approx \lambda^2/2+2$. These expressions, found by perturbation theory, are in excellent agreement with the results of a numerical analysis. Figure 1 shows two types of discrete solitons. Figure 1a shows an approximate solution of (12) found for a large value of λ.

In the limit of small values of λ, the solution tends toward the ground state of the continuum problem: a wide distribution including many modes. Figure 1b shows a discrete soliton of this sort, corresponding to the value $\lambda = 0.02$.

The factor which primarily determines the physical significance of discrete solitons is their stability. As in the continuum case, some reliable qualitative conclusions can be drawn from an analysis of an important characteristic of a nonlinear system: the dependence of P on the parameter λ (Refs. 8 and 26). Figure 2 shows curves of $P(\lambda)$ for several values of the size of the discrete file, $N=M$; large values of λ correspond to narrow, large-amplitude distributions. We see that, beginning at $\lambda \approx 1$, $P(\lambda)$ is essentially the same as the asymptotic result derived above and is independent of N. As N increases, the maximum of the distribution in the region of small values of λ approaches a constant corresponding to the continuum case. It was shown in Ref. 8 that the sign of the expression $dP/d\lambda$ plays a decisive role in the stability of solitons in the case of the 1D discrete nonlinear Schrödinger equation. We believe that the powerful method for studying stability which was developed by Laedke *et al.*[8] can also be applied to 2D systems. This assertion is supported by a numerical simulation of the dynamics of discrete solitons which we have carried out. Solitons corresponding to a positive sign of $dP/d\lambda$ exhibit a tendency toward a stable dynamics, in contrast with negative solitons. The situation is illustrated by Figs. 1 and 3. Figure 1 shows a steady-state solution, which remains of the

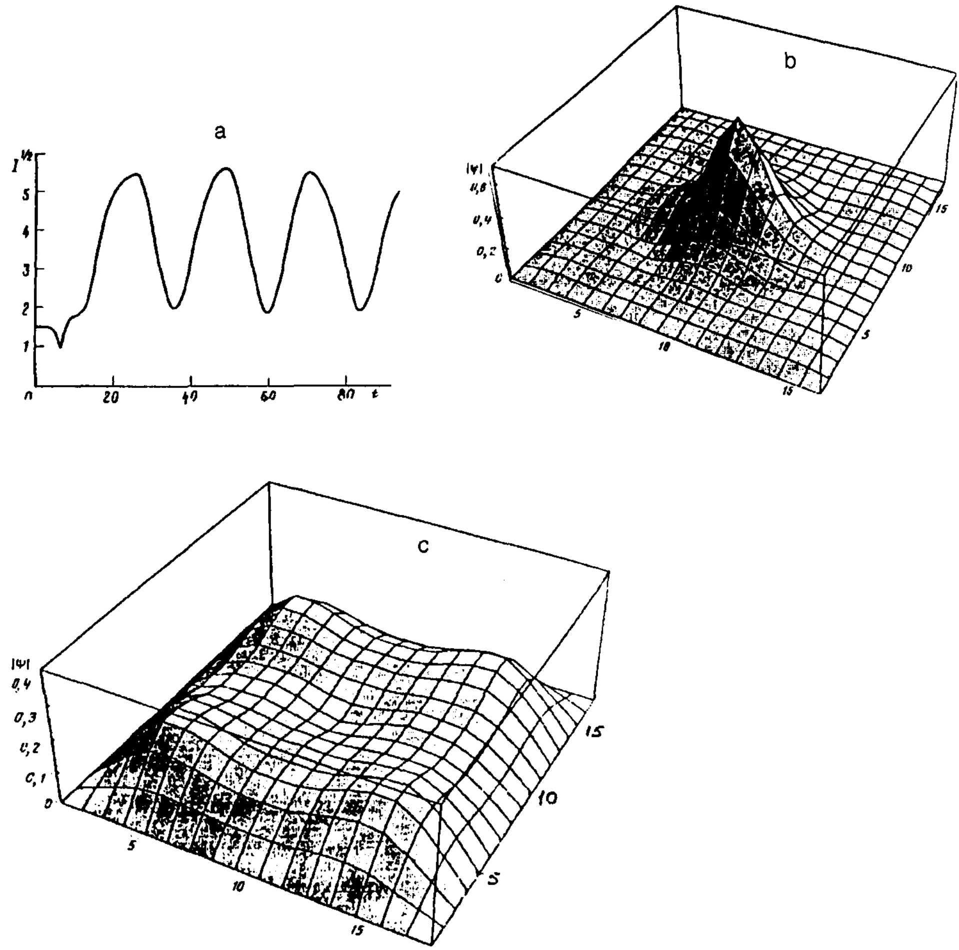

FIG. 3. Onset of an instability of a discrete 2D soliton. a—Time evolution of the "virial" $I=\sum_{m=-N/2,n=-N/2}^{m=N/2,n=N/2}(m^2+n^2)|\Psi_{mn}|^2/P$; b—state corresponding to the minimum value of I; c—state corresponding to the maximum value of I.

same form, without any changes, up to a time $t=40$. Figure 3 demonstrates the onset of an instability of a soliton in the region of decreasing $P(\lambda)$. We see that the nonlinear stage of the instability leads to the formation of "breather" or "pulson" entities. The nontrivial dynamics of such structures over long time intervals will be examined in the future.

An important new feature which stems from the discrete nature of the system is the coexistence of stable solitons and unstable states. Interestingly, the effects which stem from the discrete nature of the system stabilize the ground state of the corresponding continuum problem (this ground state is known to be slightly unstable).

In summary, we have analyzed the structure of the ground states of a 2D discrete nonlinear Schrödinger equation. We have found how effects stemming from the discrete nature of the system stop the collapse which occurs in the continuum case. The instability of a wide initial distribution in the nonlinear stage leads to a localization of energy in a few modes over a finite time. This process may be the mechanism for the formation of narrow, large-amplitude states in multidimensional discrete systems.

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