On anomalously slow stochastization in certain two-dimensional field theory models

V. E. Zakharov, M. F. Ivanov, and L. N. Shur
L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences

(Submitted 17 May 1979)
Pis'ma Zh. Eksp. Teor. Fiz. **30**, No. 1, 39–44 (5 July 1979)

The problem of stochastization of the initial conditions is studied numerically within the framework of the nonlinear Klein–Gordon equation. A specific case of the classical Yang–Mills equations is shown to identify the absence of stochastization.

PACS numbers: 11.10.Lm, 03.50.Kl

1. Recent years have witnessed a rapid development of methods of exact integration of nonlinear equations that occur naturally in physics, especially the two-dimensional relativistically invariant field theory models (see, eg. Ref. 1). However, regardless of the sufficient generality and power of these methods, they were unable to provide an answer to whether a given model is integrable. To answer this question within the framework of purely analytical models is hardly possible. In this paper, we wish to emphasize the fact that in solving this problem methods of numerical modeling by an electronic computer may be successfully applied.

We studied two-dimensional field theory models described by the nonlinear Klein–Gordon equations

 0021-3640/80/010034-06$00.60

$$U_{tt} - U_{xx} + U^3 = 0. \qquad (1)$$

Current analytical methods provide for the integration of Eq. (1) in two cases

$$F(U) = c_1\exp(\lambda x) + c_2\exp(-\lambda x) \qquad [2]$$

and

$$F(U) = c_1\exp(\lambda x) + c_2\exp(-2\lambda x) \qquad [3]^{1)},$$

(c_1, c_2, λ are arbitrary constants). It may be shown that in only these two cases the system [Eq. (1)] has nontrivial (different from momentum and energy) integrals as follows

$$I = \int f(U, U_t, U_x, U_{xt}, U_{xx}, \ldots)dx. \qquad (2)$$

In our experiments, $F(U)$ was a polynomial in the odd degrees of U. Models of this kind are frequently used in field theory. Thus, with regard to the problem of a spontaneous perturbation of the vacuum symmetry, a number of works considered the Higgs field[4]

$$F(U) = -m^2 U + U^3.$$

The question of integrability of the classical Yang–Mills field is exceptionally important for classical and quantum field theory

$$[\nabla_i , F_{ik}] = 0, \qquad F_{ik} = [\nabla_i , \nabla_k], \qquad (3)$$

An indirect indication of the possibility of its integrability is the fact that the particular solutions of Eq. (3) in Euclidean space-time care given by self-dual equation.

$$F_{ik} = \pm \frac{1}{2} \epsilon_{iklm} F_{lm} ,$$

which constitutes an integrable system.[5] In a simpler case of a SU_2 gauge group where A_i may be assumed as three-dimensional isotropic vectors, Eq. (3) may be reduced to the form of Eq. (1) by means of the following substitution:

$$\frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_4} = 0, \quad A_1 = A_2 = 0, \quad A_3 = nU, \quad A_4 = mU,$$

$$n^2 = m^2 = 1, \quad (nm) = 0 \qquad (4)$$

moreover

$$U_{tt} - U_{xx} + U^3 = 0 \qquad (5)$$

2. In those cases where the solutions to Eq. (1) are solitons, the question of

integrability may be decided, to some degree, by the numerical modeling of soliton collisions. (Inability to integrate is indicated by inelasticity of collisions.) In the case of the Higgs field, inelastic collisions were shown in Ref. 8). However, Eq. (5) has no soliton-type solutions. Therefore, to clarify the integrability question of it is reasonable

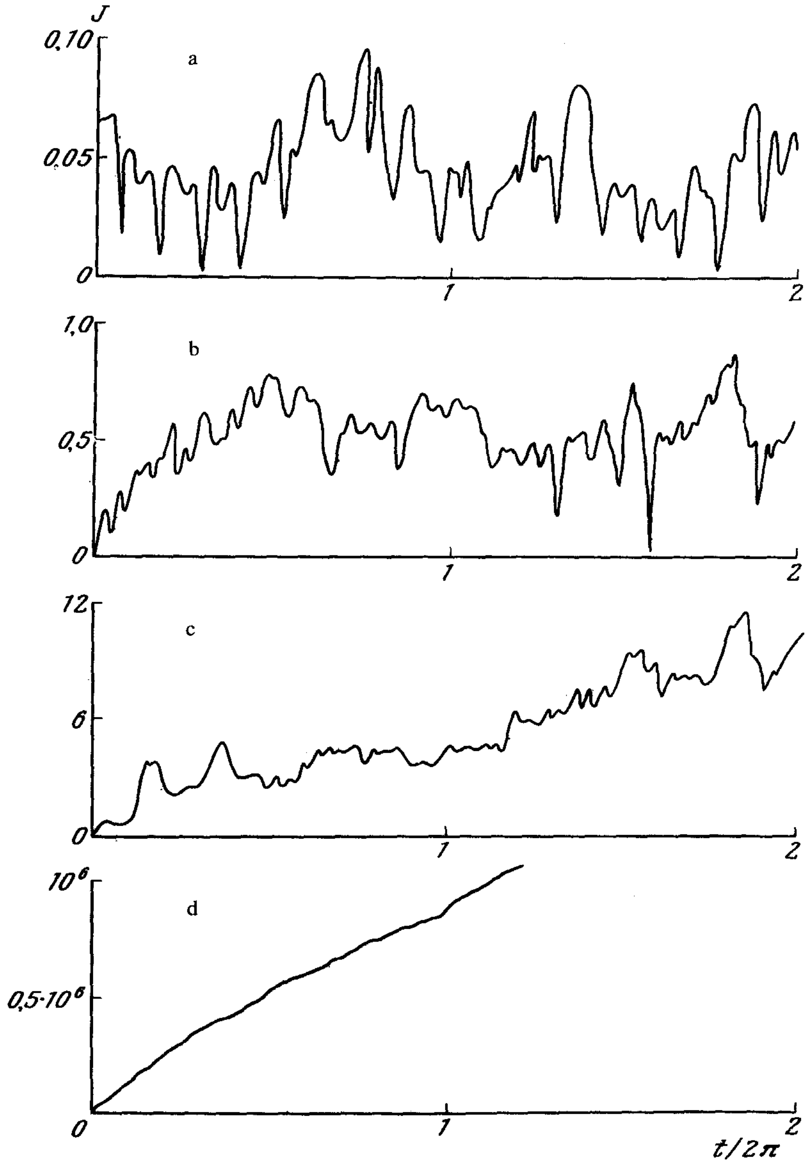

FIG. 1. Dependence of function J on time for equations (a) $U_{tt} - U_{xx} + 10\sin U = 0$; (b) $U_{tt} - U_{xx} + 10U^3 = 0$; (c) $U_{tt} - U_{xx} + 10(U - 0.6U^3 + 0.1U^5) = 0$; (d) $U_{tt} - U_{xx} + 10U(U^2 - 0.1089)\times(U^2 - 0.36)(U^2 - 0.81) = 0$; initial condition is shown in Fig. 2e.

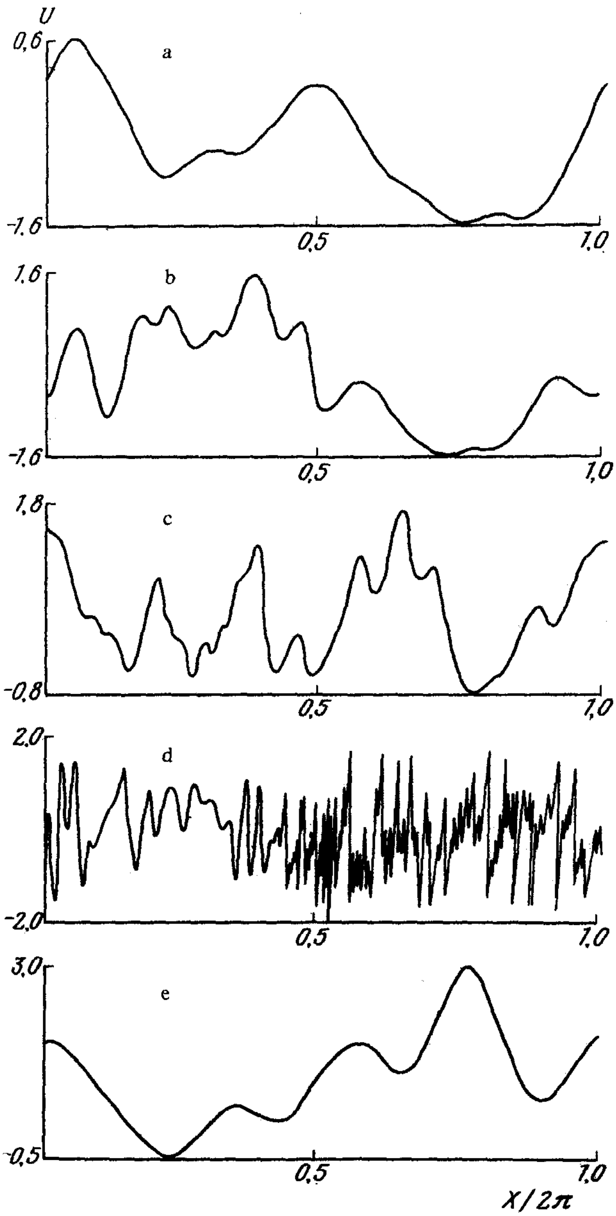

FIG. 2. Form of solutions at a time $t = 3\pi$ for equations shown in the caption of Fig. 1. (2a–2d), curve in 2e represents the initial condition.

to consider the problem of stochastization of an arbitrary initial condition of Eq. (1) under periodic boundary conditions.

This formulation approaches the classical work of Fermi, Past and Ulam.[6] The Friedman principles of statistical mechanics indicate that an evolution of a sufficiently general initial condition should exhibit a tendency toward a equipartition of energy with respect to the degrees of freedom. In the language of the X Fourier harmonics, this means that the energy should flow into a region of large wave numbers K. More-

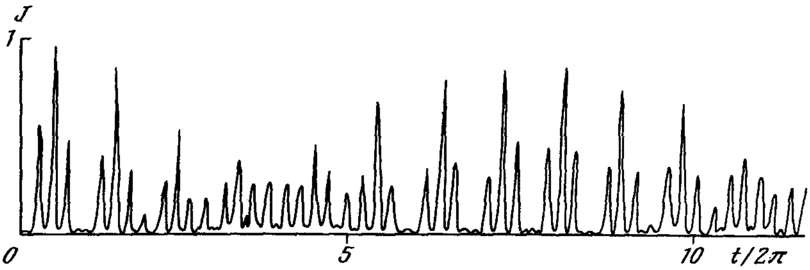

FIG. 3. Dependence of function J on time for Eq. (5) for the initial condition $U_0(X) = 0.8 + 0.2 \cos X$.

over, the energy distribution function in the k-space E_k should converge to a constant (due to finiteness of the total energy), and $J = \Sigma_k k^2 E_k^2$ should grow indefinitely. The smoothness of the function $U(x)$ in this case will deteriorate sharply and a tendency for stochastization—conversion of the function to white noise—is demonstrated. The conservation of the smoothness of $U(x)$, as well as conservation of the mean value of J during a relatively long period of time, definitely point to the existence in Eq. (1) of implicit integrals of motion and, possibly, to their integrability. An additional indication of integrability is the quasi-periodic behavior of E_k in the case where the initial condition contains a small number of Fourier harmonics.

Inasmuch as the classification of integrable systems of the Eq. (1)—type is far from completion, and because Eq. (1) may also have particular quasi-periodic solutions also in the nonintegrable cases, the initial condition

$$U_0(x) = U(x, t)\big|_{t=0}$$

should be of a sufficiently general form. Otherwise, there is a risk of being close to the quasi-periodic motion, where the conservation of the value of J would demonstrate once more the correctness of the Kolmogorov–Arnol'd–Moser theory (see, e.g., Ref. 7).

We solved Eq. (1) for the following periodic boundary conditions

$$U\big|_{x=0} = U\big|_{x=2\pi}, \quad U_x\big|_{x=0} = U_x\big|_{x=2\pi}.$$

As the initial condition $U_0 = U\,|_{t=0}$, we picked a segment of a trigonometric series

$$U_0(x) = U_0 + \Sigma a_n \cos(nx + \phi_n),$$

where U_0 and a_n are random numbers in the interval (0,1), and ϕ_n, random numbers in $(0,2\pi)$. Figure 2e shows the characteristic curve of the function $U_0(x)$. Figures 1 and 2 show curves of $J(t)$ and a form of solution of $U(x)$ at $t = 3\pi$ for four different forms of the function $F(U)$. Figures 1a and 2a correspond to the integrable case "sine-Gordon." Figures 1b and 2b correspond to the Yang–Mills problem $F(U) = 10\,U^3$. Figures 1c and 2c correspond to the polynomial $F(U) = 10(U - 0.6U^3 + 0.1U^5)$ which contains no zeros, except $U = 0$. Figures 1d and 2d correspond to the polynomial $F(U) = 10U(U^2 - 0.1089)(U^2 - 0.36)(U^2 - 0.81)$ which contains three zeros for $U > 0$. Figure 3 shows a curve of $J(t)$ for Eq. (5) for the following initial condition

$$U_0 = 0.8 + 0.2\cos X.$$

The results of numerical experiments show that the general polynomial $F(U)$ exhibits a tendency to stochastize and set a nonintegrable system. This tendency is rapidly enhanced if $F(U)$ contains additional zeros, a fact explained by the existence of unstable stationary states in the system. On the other hand, Eq. (5) behaves practically the same way as the integrable "sine-Gordon" system. This result permits us to hope that Eq. (5) [and, possibly, the entire system of the Yang–Mills equations, Eq. (3)] contains implicit integrals of motion of a more complex form that Eq. (2), and constitutes an integrable system.

In conclusion, the authors wish to thank S.V. Manakov and Ya.G. Sinaya for taking an active part in the discussion of this work at all its stages.

[1]V.E. Zakharov and A.V. Mikhaĭlov, Zh. Eksp. Teor. Fiz. **74**, 1953 (1978) [Sov. Phys. JETP **47**, 1017 (1978)].
[2]M.J. Ablowitz, D.S. Kaup *et al.*, Phys. Rev. Lett. **30**, 1262 (1973).
[3]A.B. Shabat, Dokl. Akad. Nauk SSSR (in press).
[4]P. Higgs, Phys. Lett. **12**, 132 (1966).
[5]A.A. Belavin and V.E. Zakharov, Pis'ma Zh. Eksp. Teor. Fiz. **25**, 603 (1977). [JETP Lett. **25**, 567 (1977)].
[6]E. Fermi, Nauchnye trudy (Scientific works). M., II (1972).
[7]V.I. Arnol'd, Usp. Mat. Nauk **18**, 13 (1963).
[8]V.G. Makhankov, Physics Reports **35**, 1 (1978).