Effect of a gravitational field on the motion and radiation of relativistic particles in a magnetic field and in the electric field parallel to it

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It is shown that the increase in the energy of charged particles in a strong gravitational field and in parallel magnetic and electric fields can be limited, that in special relativity their energy cannot increase simultaneously with the transverse component of the velocity, and that the maximum magnetic bremsstrahlung can change its direction with time.

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At present, astrophysical objects such as neutron stars are known which have a strong gravitational field whose dimensionless potential $\psi = -GM/c^2R_0 \approx -0.1$. These objects also have a strong magnetic field **H**. If the magnetic axis does not coincide wih the axis of rotation or if the magnetic field varies for other reasons, then an electric field **E**, which accelerates charged particles, will appear.

Under these conditions a number of qualitative mechanical and electrodynamic relativistic effects occur in a strong gravitational field. We shall examine them in the post-Newtonian approximation, i.e., linear in ψ and in its derivatives.

Since the detailed structure of these fields is not known, we shall examine a simple model, which makes it possible to obtain the mentioned effects by avoiding cumbersome calculations. Let us assume that the **E**, **H**, and $\nabla\psi$ vectors are parallel to each other and vary over a much greater distance than the characteristic length of acceleration of the particles and their Larmor radius. Therefore, these vectors will be assumed constant in the equations of motion and of particle radiation, and their dependence on time and distance will enter into the calculation as a slowly varying parameter.

1. Acceleration of particles in the electric field. In the approximation indicated above, the time and space components of the equation of motion of a particle have the form

$$\frac{d\Gamma}{dt} + \Gamma\left(\frac{\partial\psi}{\partial t} + 2\mathbf{v}\nabla\psi - \frac{v^2}{c^2}\frac{\partial\psi}{\partial t}\right) = \frac{e}{mc^2}(1-2\psi)\mathbf{E}\times\mathbf{v}\,, \tag{1}$$

$$\frac{d(\Gamma\mathbf{v})}{dt} - \Gamma\left(2\mathbf{v}\frac{\partial\psi}{\partial t} + 2\mathbf{v}(\mathbf{v}\nabla\psi) - (c^2+v^2)\nabla\psi\right) = \frac{(1+2\psi)e}{m}\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{H}}{c}\right). \tag{2}$$

Here m and e are the mass and charge of the particle; $\mathbf{v}$ is its velocity, whose components are dx^α/dt (t is the time in the system of a distant observer); $\mathbf{E}$ and $\mathbf{H}$ are vectors with the components E_α and H_α, respectively; and

$$v^2 \equiv \left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt}\right)^2;\ \Gamma = 1/\sqrt{1 - \frac{v^2}{c^2} + 2\psi(1+v^2/c^2)}. \tag{3}$$

We readily infer from Eq. (1) that for $E=0$ and $\partial\psi/\partial t=0$ the quantity $(1+2\psi)\Gamma$ will be conserved and may be regarded as the dimensionless energy of the particle. From Eq. (3) we obtain the relation

$$\beta^2 \equiv v^2/c^2 = 1 + 4\psi - (1+2\psi)/\Gamma^2 \to 1 + 4\psi$$

for the relativistic particles ($\Gamma \gg 1$); when $\psi = -0.1$, $\beta_{max} = 0.6$.

Setting $v \approx c$ in Eq. (1), we obtain

$$\frac{d\Gamma}{dt} + 2c\Gamma|\nabla\psi| = eE/mc.$$

Hence

$$\Gamma = \frac{eE}{2mc^2|\nabla\psi|} + [\Gamma_0 - eE/(2mc^2|\nabla\psi|)]\exp(-2c|\nabla\psi|t)\,, \tag{4}$$

where Γ_0 is the initial value of Γ. This solution is valid under two conditions. First, the characteristic time $1/(c\nabla\psi)$ must be short compared to the variation time of the fields E and $\nabla\psi$. This condition is satisfied for known objects. Second, the relation $E/\nabla\psi$ must also vary sufficiently slowly. If the electric field is produced in the same way as the field of the oblique rotator, then it and $\nabla\psi$ will vary as $1/R^2$, so the second condition is also satisfied. Equation (4) shows that, in contrast to the special theory of relativity, the energy of the particle, rather than increasing indefinitely, will approach a finite limit

$$\epsilon_{max} = mc^2\Gamma_{max} = eE/2\nabla\psi.$$

If we assume that $E \approx 10^6$ (cgs) and $\nabla\psi \approx 10^{-7}\ \psi \approx 10^{-8}$ cm^{-1}, then $\epsilon_{max} \approx 2\times10^{14}$ erg. The maximum energy is independent of the particle mass.

2. Particle motion in the magnetic field and in the electric field parallel to it. If both fields are directed along the z axis, then for the complex quantity $v_+ = v_x + iv_y$ we have

$$\dot{v}_+ + v_+\left[eE/(mc\,\Gamma) - 4c\,|\nabla\psi| - 2\frac{\partial\psi}{\partial t}\right] = -i\frac{\Omega}{\Gamma}v_+ ,$$

where Ω is the cyclotron frequency.

$$v_+(t) = v_+(0)\exp\left[-t\left(\frac{eE}{mc\,\Gamma} - 4c\,|\nabla\psi| - 2\frac{\partial\psi}{\partial t}\right)\right]\exp\left(-i\,\Omega\,t/\Gamma\right). \quad (5)$$

Note that the situation in the general theory of relativity is qualitatively different from that in special relativity. In the latter case, the sign of the time derivative of the energy, for the conditions examined by us, is opposite to that of the transverse velocity of the particle. In fact[1]:

$$\dot{v}_\perp = \sqrt{1 - v^2/c^2}\left(\mathbf{v}\times\vec{\Omega} - \frac{e}{mc^2}\mathbf{v}_\perp(\mathbf{v}\times\mathbf{E})\right).$$

Multiplying this equation by $v_\perp$, we obtain:

$$\frac{d}{dt}v_\perp^2 = -\frac{2}{mc^2}v_\perp^2\frac{d\epsilon}{dt}\sqrt{1 - v^2/c^2}.$$

In contrast, the energy and transverse velocity in our case can increase simultaneously if $\Gamma_{max}/\Gamma < 2$. Thus, the transverse velocity of sufficiently energetic particles always increases.

3. Magnetic bremsstrahlung. The Maxwell equations for variable fields $\mathbf{E}'$ and $\mathbf{H}'$ in the post-Newtonian approximation have the form

$$\operatorname{div}\mathbf{H}' = 0;\ \operatorname{div}\mathbf{E}' = 4\pi(1-\psi)\rho;$$

$$\operatorname{rot}\mathbf{E}' = -\frac{1}{c^*}\frac{\partial\mathbf{H}'}{\partial t};\ \operatorname{rot}\mathbf{H}' = \frac{1}{c^*}\frac{\partial\mathbf{E}'}{\partial t} + \frac{4\pi}{c^*}(1-\psi)\mathbf{j}.$$

Here $c^* = (1+2\psi)c$ and $\mathbf{j} = \rho\mathbf{v}$. Setting $\mathbf{H}' = \operatorname{rot}\mathbf{A}$, we obtain

$$\mathbf{E}' = -\frac{1}{c^*}\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi.$$

Thus the equations for the potentials ϕ and $\mathbf{A}$ have the form

$$\Delta\phi - \frac{1}{c^{*2}}\frac{\partial^2\phi}{\partial t^2} = -4\pi(1+2\psi)\delta(\mathbf{r}-\mathbf{r}_c);\ \Delta\mathbf{A} - \frac{1}{c^{*2}}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}e\mathbf{v}\delta(\mathbf{r}-\mathbf{r}_c)$$

where $\mathbf{r}_c$ is the radius vector of the point charge. Using the method described in Ref. 2, we obtain an analog of the Shutt formula:

$$W = \frac{(1+4\psi)e^2\Omega^2}{c}\sum_{\nu=1}^{\infty}\nu^2\int_0^{\pi}\frac{\sin\theta\,d\theta}{(1+2\psi-\beta_{\parallel}\cos\theta)^3}\Big[\beta_{\perp}^2 J_{\nu}'^2$$

$$+\{[\beta_{\parallel}-(1+2\psi)\cos\theta]/\sin\theta\}^2 J_{\nu}^2\Big],$$

where θ is the angle between the direction of propagation and the field **H**, ν is the number of the harmonic, and the argument of the Bessel function and its derivative is

$$\nu\beta_{\perp}/(1+2\psi-\beta_{\parallel}\cos\theta).$$

The magnetic bremsstrahlung is qualitatively different from the corresponding effect in special relativity. On the one hand, for any $\beta_{\parallel} < \beta_{max}$ the radiation is concentrated near the angle θ_m, which is determined by the relation $\cos\theta_m = \beta_{\parallel}/\beta_{max}$. (When $\beta_{\parallel} = \beta_{max}$, $\beta_{\perp} = 0$ and the radiation will be absent.) Thus, the "sharpness" of the diagram is also preserved in general relativity. On the other hand, we have seen [Eq. (5)] that $\beta_{\perp}$ varies slowly with time, and hence the angle θ_m corresponding to the radiation maximum must also vary. Since the sign of variation of $\beta_{\perp}$ depends on the particle energy, the direction of maximum radiation is different for particles of different energies. This effect is practically absent in special relativity because the transverse velocity of the particle and hence its radiation decrease with increasing energy.

[1]L.D. Landau and E.M. Lifshitz, Teoriya polya (Field Theory), Nauka, M. 1973.
[2]A.A. Sokolov and I.M. Ternov, Relyativistskiĭ electron (Relativistic Electron), Nauka, M., 1974.