# Particle conductivity in a two-dimensional random potential

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It is shown that the conductivity in a two-dimensional system with disorder at low frequencies depends logarithmically on the frequency. The connection between this result and the conclusions of Ref. 1 about the total localization in the two-dimensional case is discussed.

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This work was stimulated by a paper[1] in which it was asserted that all single-particle states in the defect field in the two-dimensional case are localized. We shall investigate the quantum corrections to the diffusion law or to the conductivity of an infinite system as a function of the frequency $\omega$.

The conductivity $\sigma$ can be expressed in terms of the Green's functions of the Schrödinger equation with a random potential[2]:

$$\sigma = e^2 \int dE \nu(E) \frac{n_E - n_{E+\omega}}{\omega} D(\omega), \tag{1}$$

$$D(\omega) = \frac{1}{d} \int dr < \mathbf{v}(0) G^R_{E+\omega}(0, r) \mathbf{v}(\mathbf{r}) G^A_E(\mathbf{r}, 0) >, \tag{2}$$

where $D_E(\omega) \equiv D(\omega)$ is the diffusion coefficient of a particle with an energy $E$, $d$ is the dimension of the space, $G^R$ and $G^A$ are the lagging and leading Green's functions, respectively, and the angle brackets $\langle \quad \rangle$ denote averaging over the possible locations of the impurities. The energy $E$ is assumed to be large compared with the reciprocal time of the mean free path, $\hbar/\tau$. We shall use the "cross technique" of averaging[2] to calculate the average values. As is well known, for $E\tau/\hbar \gg 1$ the diagrams with crossed dashed lines (for example, the plot in Fig. 1a) are small. Disregarding these diagrams, we obtain $D = D_0 = v^2\tau/d$ (for simplicity, the scattering is assumed to be isotropic $\tau = \tau_{tr}$).

To calculate the quantum corrections, we must examine the diagrams in Fig. 1. Such diagrams were earlier examined by Langer and Neal,[3] who showed that each

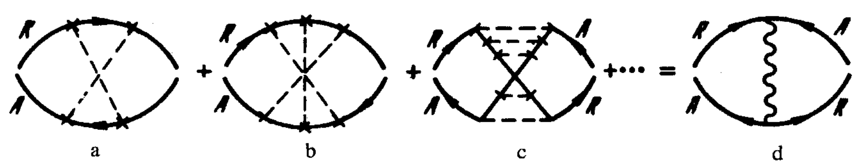


FIG. 1.

  

individual diagram gives a small correction to the diffusion coefficient. It is important that the *sum* of all the ladder-type diagrams, which is illustrated in Fig. 1d by a wavy line, has a characteristic diffusion pole

$$C(\mathbf{q}, \omega) = \frac{1}{\tau^2}\frac{1}{Dq^2 - i\omega} - \frac{1}{\tau}; \quad \mathbf{q} = \mathbf{p} + \mathbf{p}_1, \quad qv\tau << 1, \quad \omega\tau << 1. \tag{3}$$

This peculiarity of the "fan-shaped" ladder coincides in total momentum with the well-known property of the ordinary diffusion ladder

$$< \rho, \rho >_{\omega,\mathbf{q}} \sim \frac{1}{Dq^2 - i\omega} \tag{4}$$

as a function of the transferred momentum $\mathbf{q}$. Such a coincidence is a consequence of the invariance with respect to time reversal as a result of transformation of the wave functions $\psi(p) \rightarrow \psi^*(-p)$. Henceforth, the wavy line in the diagrams will denote both the ordinary diffusion ladder and the fan-shaped ladder.

In diagram 1*d* we examine integration over the momenta $\mathbf{q}$ of the wavy line. The singular frequency contribution comes from the region of small $\mathbf{q}$. Therefore, $\mathbf{p}_1 \approx -\mathbf{p}$ and the singular quantum correction to the conductivity is negative. After calculation we obtain[1)]

$$D = D_0 \left\{ 1 - \frac{3\hbar^2\pi}{2SmE\sqrt{6\omega\tau}} \right\}, \qquad d = 1 \tag{5}$$

$$D = D_0 \left\{ 1 - \frac{\hbar}{2\pi E\tau} \ln\frac{1}{\omega\tau} \right\}, \qquad d = 2 \tag{6}$$

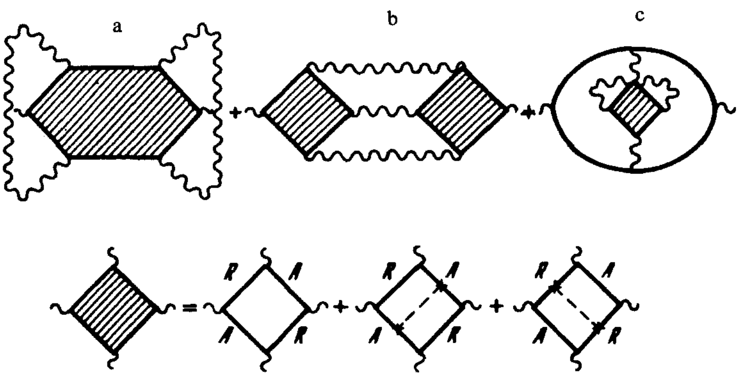


FIG. 2.

$$D = D_0 \left\{ 1 + \frac{3\hbar^2\sqrt{6\omega\tau}}{16E^2\tau^2} \right\}. \qquad d = 3 \tag{7}$$

For the one-dimensional case the localization length is of the order of the mean free path $l \sim v\tau$ and the diffusion region $ql \ll 1$ is missing. Therefore, Eq. (5) will apply to a wire of cross section $S \gg \hbar^2 p^{-2}$, for which a diffusion region exists,[6] and the second term in the braces is a small correction.

For simplicity, we have explicitly calculated the conductivity. The quantum correction for the correlator $\langle\rho\rho\rangle$ can also be determined. We find that the diffusion equation (4) remains true and the diffusion coefficient is in fact given by Eqs. (5)–(7).

Of particular interest is the two-dimensional case, in which the correction to the conductivity increases logarithmically with decreasing frequency, and hence a question arises about the summation of the set of principal logarithms of the type $[(\hbar/E\tau)\ln(1/\omega\tau)]^n$. The diagrams giving contributions of order $[(\hbar/E\tau)\ln(1/\omega\tau)]^n$. are illustrated in Fig. 2. The number of different diagrams and the complexity of their structure increase dramatically with increasing order. In each order and for any block diagram $[(\hbar/E\tau)\ln(1/\omega\tau)]^n$. coincides with the number of integrations over the momenta of the wavy lines. It was found, moreover, that in second order the diagrams in Fig. 2b cancel each other and the contribution such as the diagram in Fig. 2a is cancelled by the contribution from the diagram in Fig. 2c. In other words, the term $[(\hbar/E\tau)\ln(1/\omega\tau)]^2$. in Eq. (6) is missing.

We can assume that this result, rather than being accidental, is a consequence of a renormalized invariance.[6] In fact, for $\omega\tau \ll 1$ and $q = 0$ there are no parameters with dimensions of length or frequency, and there is only one dimensionless parameter $R_0 = 1/E\tau$, which plays the role of a bare interaction constant and coincides with the resistance of a square of the film for $\omega\tau \sim 1$, measured in units of $\pi e^{-2}\hbar \sim 13$ k$\Omega$. Therefore, if the frequency scale is changed $\omega\tau \to \omega/\omega_1$, it can be compensated for by a corresponding change of concentration of the impurities according to the law $R_0 \to R(\omega_1)$, so that $R(\omega\tau, R_0) = R[\omega/\omega_1, R(\omega_1)]$. Differentiating this equation with respect to $\omega_1 = \omega$, we obtain an equation for the renormalized group in the form

$$\frac{dR(\omega)}{d\ln\omega} = f(R) = -\frac{R^2}{2\pi} + 0(R^3). \tag{8}$$

The solution of Eq. (8) for $R \ll 1$ shows that all terms of order $(1/E\tau)^n \ln^n (1/\omega\tau)$ with $n > 1$ are missing in the expression for $D = \hbar/mR$.

The conductivity at low frequencies can be solved by the $f(R)$ function when $R \approx 1$. In this region, however, $f(R)$ depends on the specific features of the model and to determine it is as difficult as it is to calculate $R(\omega)$ directly. Nonetheless, Eq. (8) can be used for a qualitative interpolation between the regions of small and large $\omega$. If the disorder is extremely large, then the states of the particle are localized, and hence $f(R)$ increases when $R \gg 1$ (for example, $f = -2R$ for $R \sim \omega^{-2}$). If $f(R)$ has no zeros at finite $R$, then $R(\omega)$ decreases to zero as $\omega \rightarrow 0$. Thus, in the two-dimensional case all the states are localized at arbitrarily low concentrations of the impurities. If $f(R)$ vanishes at a certain point $R_1$, then $R(\omega) \rightarrow R_1$ as $\omega \rightarrow 0$. Generally, the vanishing of the static conductivity is not equivalent to localization, which corresponds[7] to a density correlator of the form

$$< \rho\rho >_{\omega,\mathbf{q}} = \frac{A(\mathbf{q})}{\omega} . \tag{9}$$

If, however, a single invariant charge $R$ exists in the scale transformations, then the small $R$'s correspond to a large disorder and hence to localization.[8]

Thus our results are very similar to those of Abrahams *et al.*,[1] although the details of their calculations have not been published. The difference is that the variable in the renormalized group of Abrahams *et al.*[1] is the sample size $L$ rather than the frequency $\omega$. For a sample of finite size it is important to know the method used to determine the conductivity and the frequencies in question. In a finite-size sample absorption occurs at frequencies $\omega > \hbar/mL^2$. On the other hand, if $\omega \gg DL^{-2}$, then the logarithms in the quantum corrections are independent of $L$. In other words, if $R \ll 1$, then there exists a frequency region $DL^{-2} \gg \omega \gg \hbar/mL^2$ for which the quantum corrections are given by Eq. (6) in with $(L/l)^2$ is substituted for $1/\omega\tau$ in the argument of the logarithm. Therefore, we can assume[1] that

$$\frac{dR}{d \ln L^{-2}} = \beta(R) = \frac{1}{2\pi} R^2 + 0(R^3). \tag{10}$$

If the sample is connected in a circuit, then the discrete levels will have a width of the order of $\gamma \sim \hbar DL^{-2}$. Therefore, the frequency can be set equal to zero. The divergence in this case is cut by $\gamma$ and we can again attempt to write Eq. (10). It is important to note that the function $f$ coincides with the function $\beta$ only when $R \ll 1$, i.e., in the approximation of the principal logarithms, and when $R \gg 1$ $f$ is different from $\beta$. Everything said about the behavior of the function $f$ at small $R$ evidently applies to the function $\beta$.

The conductivity in a two-dimensional disordered system was numerically modeled in Refs. 9 and 10, in which the level shift in a finite sample due to a change of the boundary conditions was calculated. This quantity, which is useful for determining the onset of localization in numerical experiments, is equal in order of magnitude to the conductivity. However, the exact relation between these quantities is not known. Our conclusions coincide better with those of Ref. 9.

In conclusion, we note that the energy-relaxation processes cut off the logarithms in Eq. (6) at an appropriate parameter $1/\tau_\epsilon$ if $\omega\tau_\epsilon \ll 1$.

[1)]The importance of taking into account the diffusion poles in the calculation of the quantum corrections for the conductivity was pointed out for the first time by Maleev and Totrverg[4]; however, they did not examine the fan-shaped diagrams and hence the corrections calculated by them have a smallness $\omega/E$ that complements Eqs. (5)–(7).

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