Stability of weak shock waves

E. A. Kuznetsov, M. D. Spektor, and G. E. Fal'kovich
Institute of Automation and Electrometry, USSR Academy of Sciences, Siberian Branch

(Submitted 12 July 1979)
Pis'ma Zh. Eksp. Teor. Fiz. **30**, No. 6, 328–330 (20 September 1979)

It is shown that a weak shock wave is unstable relative to transverse modulations.

PACS numbers: 47.40.Nm

1. Until now, the stability of shock waves was investigated in terms of ideal gas dynamics. As is well known,[1] explosions in this case are stable against transverse perturbations. The question remains open, however, whether the shock waves with a finite width wave front are stable. This problem is also of interest because it is directly connected with strong acoustic turbulence. The two current approaches to this problem predict different types of turbulence spectra: $\epsilon_k \sim k^{-2}$ [2] and $\epsilon_k \sim k^{-3/2}$.[3]

According to the first point of view, sound turbulence is an aggregate of weakly interacting shock waves and its ϵ_k spectrum is determined primarily by explosions. An alternative point of view attributes an important part to the interaction of waves propagating at an angle of $\theta \sim (v/c_s)^{1/2}$ to each other (here v is the perturbation rate and c_s is the velocity of sound). This process leads to isotropization of the spectrum.

In this paper we show that weak shock waves are unstable relative to transverse perturbations and that they have a maximum increment in the range of angles $\theta \sim (v/c_s)^{1/2}$.

2. As is well known,[4] one-dimensional propagation of small-amplitude sound waves is described by Burgers equation

$$u_t + uu_x - \mu u_{xx} = 0,$$

where μ is the coefficient of "viscosity." This equation has a solution in the form of a shock wave that propagates with a velocity v,

$$u_0 (x - vt) = v\left[1 - \mathrm{th}\, \frac{v}{2\mu}(x - vt)\right] \tag{1}$$

with a jump of $2v$, and with a wave front $l = 2\mu/v$. Allowance for the weak transverse modulation of such waves gives the equation

$$\frac{\partial}{\partial x}(u_t + uu_x - \mu u_{xx}) = -\frac{c_s}{2}\Delta_\perp u, \tag{2}$$

which is analogous to the Kadomtsev–Petviashvili equation for multidimensional waves in a weakly dispersive medium. We examine the stability of shock waves in terms of this approximation.

Linearization Eq. (2) after a steady-state solution of Eq. (1), we obtain

$$\frac{\partial}{\partial x}[\delta u_t + u_0 \delta u_x + \delta u u_{0x} - \mu \delta u_{xx}] = -\frac{c_s}{2}\Delta_\perp \delta u .$$

Let us now go over to dimensionless arguments $x \rightarrow (x - vt)\, v/2\mu,\ t \rightarrow t\, v^2/4$

 0021-3640/80/180303-03$00.60 © 1980 American Institute of Physics

and introduce a new variable $\psi = \mathrm{ch}x\delta u$. For perturbations of the type $\psi(x)\ e^{-Et + ikr_\perp}$ we have the following spectral problem for a third-order, non-self-adjoint operator L:

$$L\psi = \mathrm{ch}\,x \frac{\partial}{\partial x} \frac{1}{\mathrm{ch}x} \left[-\frac{\partial^2}{\partial x^2} - \frac{2}{\mathrm{ch}^2 x} + (1-E) \right] \psi = \alpha k^2 \psi, \quad (3)$$

where $\alpha = 4c_s\mu^2/v^3$. At $k = 0$ this spectral problem reduces to determination of the spectrum of the Schrödinger operator

$$\left(-\frac{\partial^2}{\partial x^2} + 1 - \frac{2}{\mathrm{ch}^2 x} - E\right)\psi = 0$$

with a reflectionless potential. This operator has one discrete level $E = 0$ with an eigenfunction $\psi_0 = 1/\mathrm{ch}\,x$ and a continuous spectrum $E = 1 + p^2 > 0$

$$\psi_p = -\frac{1}{1 - ip}\, \mathrm{ch}\,x \frac{\partial}{\partial x} \frac{e^{ipx}}{\mathrm{ch}\,x}.$$

Thus, the shock wave is stable against one-dimensional perturbations with $k = 0$. Here, a perturbation of an infinitesimal shift $\delta u = \partial u_0/\partial x$ corresponds to a nondiscriminating stable eigenfunction ψ_0. At small αk^2 the stability of a shock wave is determined by the level shift $E = 0$, which can be determined from the standard scheme of the perturbation theory. For this purpose we represent ψ in the form:

$$\psi = c_0\psi_0 + \int_{-\infty}^{\infty} c_p \psi_p\, dp.$$

We also introduce the $\tilde{\psi}$ functions of the conjugate problem $(L + \tilde{\psi} = 0)$, which are connected with ψ by the relations:

$$\tilde{\psi}_0 = -x\psi, \quad \tilde{\psi}_p = -\frac{e^{ipx}}{1 - ip} - \frac{2ip}{p^2 + 1}\psi_p.$$

The equations for the c_0 and c_p coefficients follow directly from Eq. (3), like the condition of orthogonality for the eigenfunctions $\tilde{\psi}_0$ and $\tilde{\psi}_p$:

$$-Ec_0 = \frac{\alpha k^2}{2} \int_{-\infty}^{\infty} c_p \langle \tilde{0} | p \rangle\, dp,$$

$$(p^2 + 1 - E)c_p = \frac{\alpha k^2}{2\pi}\left[c_0 \langle \tilde{p} | 0 \rangle + \int_{-\infty} c_p \langle \tilde{p} | p' \rangle\, dp'\right],$$

where the matrix elements

$$\langle \tilde{p} | 0 \rangle = \langle \tilde{0} | p \rangle^* = -\frac{\pi}{1 + ip} \frac{1}{\mathrm{ch}(\pi p/2)}.$$

Hence, the correction for the level $E = 0$ appears in the second-order perturbation theory:

$$E^{(2)} = -\frac{(\alpha k^2)^2}{4\pi}\int_{-\infty}^{\infty}\frac{|\langle\widetilde{0}|p\rangle|^2}{p^2+1}\,dp = -\frac{(\alpha k^2)^2}{4}\left[\zeta(3)+\frac{\pi^2}{6}\right] < 0. \quad (4)$$

Thus, a weak shock wave is unstable against transverse modulations. It follows from Eq. (4) that the increment of this instability increases with increasing k and reaches a maximum in the region $\alpha k^2 \sim 1$ or $(kl)^2 \sim v/c_s$. The maximum is proportional to v^2/μ, i.e., it is of the order of the reversal time. Basically, the unstable perturbations are localized at the wave front. Therefore, the development of this instability produces a turbulent shock wave front.

In conclusion, the authors thank I.B. Khriplovich for a useful discussion.

[1]L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Mechanics of Continuous Media), GITTL, M., 1953.

[2]B. B. Kadomtsev and V. I. Petviashvili, Dokl. Akad. Nauk SSSR **208**, 794 (1973) [Sov. Phys. Dokl. **18**, 115 (1973)].

[3]V. E. Zakharov and R. Z. Sagdeev, Dokl. Akad. Nauk SSSR **192**, 297 (1970) [Sov. Phys. Dokl. **15**, 439 (1970)].

[4]V. I. Karpman, Nelineinye volny v dispergiruyushchikh sredakh (Non-linear Waves in Dispersive Media), Nauka, M., 1973.