Integrability of a two-dimensional generalization of the Toda chain

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It is shown that a two-dimensional generalization of the Toda chain $\phi^k_{tt} - \phi^k_{xx} - 2\exp(2\phi^{k+1} - 2\phi^k) + 2\exp(2\phi^k - 2\phi^{k-1}) = 0$ and, in particular, the equation $\phi_{tt} - \phi_{xx} + 2\exp(4\phi) - 2\exp(-2\phi) = 0$ are integrable by the inverse-problem method. A group, which is noncommutative in the second case, reduces these equations.

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1. It is well known that the Toda chain

$$\phi^k_{tt} = 2\exp(2\phi^{k+1} - 2\phi^k) - 2\exp(2\phi^k - 2\phi^{k-1}) \tag{1}$$

can be integrated by using the inverse-problem method.[1] In this paper we show that a two-dimensional generalization of this model is

$$\phi^k_{tt} - \phi^k_{xx} = 2\exp(2\phi^{k+1} - 2\phi^k) - 2\exp(2\phi^k - 2\phi^{k-1}) \tag{2}$$

and in the special case the equation

$$\phi_{tt} - \phi_{xx} + 2\exp(4\phi) - 2\exp(-2\phi) = 0 \tag{3}$$

is also integrable. We note that a number of remarkable properties of Eq. (3) were known earlier: the existence of an infinite set of integrals of motion was established,[3] and the existence of an infinite set of currents that commute with Eq. (3) was confirmed.[2] We deform the compatibility condition for the system Eq. (1) (see Ref. 1) in the following way:

$$\begin{aligned} X\Psi &= (\partial_x + V + iC_1 - iC_2)\Psi = 0, \\ T\Psi &= (\partial_t + W + iC_1 + iC_2)\Psi = 0, \end{aligned} \tag{4}$$

where

$$V_{ij} = \partial_t \phi_i \delta_{ij}; \quad W_{ij} = \partial_x \phi_i \delta_{ij}; \quad C_{1ij} = C_{2ji} = C_j \delta_{i-1,j},$$
$$C_i = \exp(\phi^{i+1} - \phi^i), \tag{5}$$

where δ_{ij} is the Kronecker symbol which in the case of a closed chain consisting of N

elements is determined in the following way:

$$\delta_{ij} = \begin{cases} 1 \text{ when } i \equiv j \bmod (N) \\ 0, \text{ in the remaining cases} \end{cases}$$

The spectral parameter λ can be introduced into the compatibility conditions (4), if they are covariant under the Lorentz transformation (see Ref. 4)

$$X = \partial_x + \hat{V} + i\lambda C_1 - i\lambda^{-1}C_2,$$
$$T = \partial_t + W + i\lambda C_1 + i\lambda^{-1}C_2. \qquad (6)$$

The commutability condition for the operators X and T for all the values of the parameter λ is equivalent to Eq. (2), which makes it possible to integrate the equation by the inverse-problem method.

2. The system of equations (2) corresponds to the action minimum

$$S = \int dx dt \sum_{k=1}^{N} \left[\frac{1}{2} \phi_\mu^k \phi^{k\mu} - \exp(2\phi^k - 2\phi^{k-1}) + 1\right]. \qquad (7)$$

The spectrum of mass excitations in the vicinity of zero is

$$m_n^2 = 4\sin^2(\pi n/N), \quad n = 0, 1, \ldots, \quad N-1.$$

The zero mode can be excluded by setting $\Sigma\phi^k = 0$. Thus, for $N=2$, Eq. (2) reduces to

$$\phi_{tt} - \phi_{xx} + 4\,\mathrm{sh}(2\phi) = 0,$$

and for $N=3$

$$\phi^1_{tt} - \phi^1_{xx} + 2\exp(2\phi^1 - 2\phi^2) - 2\exp(-4\phi^1 - 2\phi^2) = 0,$$
$$\phi^2_{tt} - \phi^2_{xx} + 2\exp(4\phi^1 + 2\phi^2) - 2\exp(2\phi^1 - 2\phi^2) = 0. \qquad (8)$$

The system of equations (8) allows an additional reduction ($\phi^1 = -\phi^2 = \phi$); in this case the equations coincide and have the form of Eq. (3).

Equation (2) can be regarded as a difference approximation of the equation

$$u_{tt} - u_{xx} - u_{yy} + h^2 u_{yyyy} + h(u_y)^2_y = 0, \qquad (9)$$

which generalizes the Kadomtsev-Petviashvili equation which describes the nonlinear waves that propagate in both directions along y in a weakly dispersing medium (h is the dispersion length).

3. To calculate the conservation laws, we must (see Ref. 5) perform a gauge transformation in the compatibility problem (4) and (5) and go over to the polar gauge. We perform a gauge transformation to diagonalize the C_1 matrix

$$X' = U^{-1}XU, \quad T' = U^{-1}TU,$$

$$U_{\alpha\beta} = N^{-1/2}\exp(\phi^{\alpha-1})\, q^{-(\alpha-1)(\beta-1)}, \quad q = \exp\left(\frac{2\pi i}{N}\right). \qquad (10)$$

In this case, the operators X' and T' have the form

$$X'_{\alpha\beta} = \delta_{\alpha\beta}\partial_x + \partial_x\hat{\phi}_{\alpha-\beta} + \partial_t\hat{\phi}_{\alpha-\beta} - i\lambda^{-1}q^{1-\alpha}\hat{C}^2_{\alpha-\beta} + i\lambda q^{\alpha-1}\delta_{\alpha\beta}, \quad (11)$$

$$T'_{\alpha\beta} = \delta_{\alpha\beta}\partial_t + \partial_t\hat{\phi}_{\alpha-\beta} + \partial_x\hat{\phi}_{\alpha-\beta} + i\lambda^{-1}q^{1-\alpha}\hat{C}^2_{\alpha-\beta} + i\lambda q^{\alpha-1}\delta_{\alpha\beta}, \quad (12)$$

where the cap denotes finite Fourier transformation:

$$\hat{\phi}_\alpha = N^{-1}\sum_{n=1}^{N} q^{n\alpha}\phi^n, \qquad \hat{C}^2_\alpha = N^{-1}\sum_{n=1}^{N} q^{n\alpha}C^2_n. \quad (13)$$

The diagonal elements $\alpha_{ii}(\lambda)$ of the scattering matrix of the operator X are independent of t. After asymptotic expansion of the element α_{11} with respect to the inverse powers as $\lambda\to\infty$, we obtain a series I_n of conservation laws:

$$I_n = \int_{-\infty}^{\infty} f_n dx, \quad (14)$$

where f_n is determined from the recurrence relations:

$$-i\partial_x A^n_\alpha - A^{n+1}_\alpha + A^{n-1}_\alpha + \sum_{k+m=n} f_m A^k_\alpha + (\partial_x\hat{\phi}_{\alpha-\beta} + \partial_t\hat{\phi}_{\alpha-\beta})A^n_\beta$$

$$-q^{1-\alpha}\hat{C}^2_{\alpha-\beta}A^{n-1}_\beta + q^{\alpha-1}A^{n+1}_\alpha = 0, \quad (15)$$

$$A^h_\alpha = \delta_{n,0}\,\delta_{\alpha,1} \qquad \text{for } n \leqslant 0. \quad (16)$$

The second set of integrals I_{-n}, which are obtained in the expansion in zero ($\lambda\to 0$), can be obtained in a similar way. The recurrence relations for this series have the form (15) and (16), where $\partial_t\hat{\phi}_{\alpha-\beta} + \partial_x\hat{\phi}_{\alpha-\beta}$ should be replaced by $q^{\beta-\alpha}(\partial_t\hat{\phi}_{\alpha-\beta} - \partial_x\hat{\phi}_{\alpha-\beta})$.

The diagonal elements of the scattering matrix are usually independent, and the expansion of each one of them produces a set of conservation laws. The matrix elements α_{ii} calculated for the operator X' are related by a simple relation:

$$a_{i+1,\,i+1}(\lambda) = a_{ii}(q\lambda), \quad (17)$$

and, hence the integrals produced by them coincide with those already calculated.

4. The X and T operators have a special form: each matrix V, W, and $C_{1,2}$ has only N nonvanishing elements. In the general case, however, these matrices are arbitrary and have N^2 complex elements. If in the general case the initial data for the Cauchy problem have the form (5), then in the process of evolution this form is preserved. In other words, the system (2) represents a limitation of the general position system to the submanifold of the manifold of all the matrices, which is invariant with respect to a current [in this case, a subset of the matrices of the form (5)]. This constraint is called a reduction in the general position system.

Each reduction places certain constraints on the set of eigenfunctions $\{\Psi(\lambda)\}$ of the pair of commuting X,T operators. These constraints must be taken into account in the formulation of the equations for the inverse problem. The following relations are derived from the X and T operators (6):

$$Q^* X(\lambda)\, Q = X(q\lambda), \quad Q^* T(\lambda)\, Q = T(q\lambda), \tag{18}$$

where $Q_{\alpha\beta} = \delta_{\alpha\beta} q^{\alpha-1}$, and hence,

$$Q\, \Psi(q^*\lambda) = \widetilde{\Psi}(\lambda), \quad \widetilde{\Psi}(\lambda) \in \{\Psi(\lambda)\}. \tag{19}$$

Hence, we obtain the property (17) for the scattering matrix. The requirement that the $C_{1,2}$, V and W matrices must be real leads to

$$X^*(-\lambda^*) = X(\lambda), \quad T^*(-\lambda^*) = T(\lambda) \tag{20}$$

and hence,

$$\Psi^*(-\lambda^*) = \widetilde{\Psi}(\lambda), \quad \widetilde{\Psi}(\lambda) \in \{\Psi(\lambda)\}. \tag{21}$$

Transformations (19) and (21) of the set $\{\Psi\}$ form the group $G_1 \approx Z_N \times Z_2$ that is responsible for reduction. We note that G_1 acts also in the complex plane of the spectral parameter ($\lambda \to q\lambda$, $\lambda \to -\lambda^*$); it follows from Eqs. (19) and (20) that the set of spectral singularities of the operators X and T is invariant with respect to G_1.

Equation (3) is the result of an additional reduction, and is equivalent to the commutativity condition of the pair of operators

$$\partial_x + \begin{bmatrix} \phi_t & -i\lambda^{-1}e^{-\phi} & i\lambda e^{2\phi} \\ i\lambda e^{-\phi} & 0 & -i\lambda^{-1}e^{-\phi} \\ -i\lambda^{-1}e^{2\phi} & i\lambda e^{-\phi} & -\phi_t \end{bmatrix},$$

$$\partial_t + \begin{bmatrix} \phi_x & i\lambda^{-1}e^{-\phi} & i\lambda e^{2\phi} \\ i\lambda e^{-\phi} & 0 & i\lambda^{-1}e^{-\phi} \\ i\lambda^{-1}e^{2\phi} & i\lambda e^{-\phi} & -\phi_x \end{bmatrix}.$$

Additional reduction gives rise to the additional generatrix p in the reduction group

$$\Psi(\lambda) \xrightarrow{p} K(\Psi^+(\lambda^*))^{-1} = \widetilde{\Psi}(\lambda); \quad \widetilde{\Psi}(\lambda) \in \{\Psi(\lambda)\}, \quad K_{ij} = \delta_{i,-j},$$

(evidently, $\rho^2 = 1$), and expands G_1 to the group $G \approx S_3 \times Z_2$ (S_3 is a permutation group of three elements).

Thus, Eq. (2) is an example of reduction of the general position system generated by the group G_1 (rather than by the set of involutions); Eq. (3) is an example of reduction generated by a noncommutative group. Equation (2) for odd N has a reduction that generalizes Eq. (3); moreover, the reduction group G is isomorphous $\widetilde{G} \times Z_2$, where $\widetilde{G}$ is a group with two generatrices p and q that satisfy the identities $p^2 = 1$, and $q^N = 1$, $pqpq = 1$.

It is conceivable that a classification of reductions in the integrable systems is associated with the classification of allowable reduction groups whose nontrivial examples are given in this section.

In conclusion, we note that the two-dimensional generalization is allowed by the set of equations $\partial_t N_k = N_k(N_{k+1}^2 - N_{k-1}^2)$ (see Ref. 1):

$$\partial_t N_k + \partial_x W_k = N_k (N_{k+1}^2 - N_{k-1}^2),$$

$$\partial_x (N_k N_{k-1}) = N_k W_{k-1} - W_k N_{k-1}.$$

In the continuous limit, these equations change to the Kadomtsev-Petviashvili equation $(u_t + u_{xxx} + uu_x)_x = u_{yy}$.

The formulation of the equations for the inverse problem and their analysis will be published elsewhere.

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