Yang-Mills theory in the sigma-model representation

E. A. Ivanov
Joint Institute for Nuclear Research

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Interpretation of the Yang-Mills theory as a theory of spontaneous violation leads to its new representation in terms of a bilocal nonlinear σ model. The potentialities of this model are discussed.

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1. In recent years, it has been expressed that the Yang-Mills theory in suitable variables may turn out to be fully integrable.[1-3] The search for convenient nonstandard variables for this theory is also important because of the need for an adequate description of its symmetric phase, which is responsible for confinement.[4,5]

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In view of this, a new formulation of the gauge theories in this paper, which demonstrates their deep unity with the nonlinear σ-models, is useful. It is based on the fact established earlier by Ogievetskiĭ and the author[6] that any gauge theory is the result of a nonlinear realization of a definite infinite parametric group $K = G \subset \times \mathscr{P}$ with a subgroup vacuum stability $G_0 \times \mathscr{P}$, where G_0 is a global symmetry subgroup and $\mathscr{P}$ is a Poincare group (see also Ref. 7).

2. Introducing an additional coordinate—the Lorentz 4-vector y_μ, we represent the group generators for $K = G \subset \mathscr{P}$ in the form:

$$P_\mu = i\frac{\partial}{\partial y^\mu}, \quad L_{\mu\nu} = i(y_\mu \partial^y_\nu - y_\nu \partial^y_\mu), \quad Q^i_\mu = y_\mu Q^i, \dots Q^i_{\mu_1 \dots \mu_n} = y_{\mu_1} \dots y_{\mu_n} Q^i, \dots \quad . \tag{1}$$

Here P_μ and $L_{\mu\nu}$ are the $\mathscr{P}$ group generators, Q^i are the G_0 group generators, which produce the algebra of the finite-parametric group G in addition to $Q^i_\mu, \dots Q^i_{\mu 1 \dots \mu n}, \dots$ (see Ref. 6).

Equation (1) allows us to contract the infinite set of Goldstonians $b^i_\mu(x)$, $b^i_{\mu_1\mu_2}(x), \dots b^i_{\mu_1 \dots \mu_n}(x), \dots$ (the parameters of the factor space $K/G_0 \times L$) to one bilocal field $b(x, y) \equiv b^k(x, y) Q^k = \Sigma_{n \geqslant 1}\, 1/n!\, b_{\mu_1 \dots \mu_n}(x)\, y^{\mu_1} \dots y^{\mu_n}$. Under the influence of the G group realized in the cosets $K/G_0 \times L$ by left-handed translations,[6] $b(x, y)$ is transformed according to the law:

$$\exp\{ib'(x, y)\} = \exp\{i\lambda(x + y)\} \exp\{ib(x, y)\} \exp\{-i\lambda(x)\}, \tag{2}$$

where $\lambda(y) = \lambda^i(0) Q^i + \Sigma_{n \geqslant 1}\, 1/n!\, \lambda^i_{\mu_1 \dots \mu_n}(0) y^{\mu_1} \dots y^{\mu_n} Q^i$ is the generating function for the constant parameters of the G group. The covariant derivatives of the Goldstonians are turned into the bilocal Cartan form

$$\omega_\mu(x, y) = -b_\mu(x) + \sum_{n \geqslant 1} \frac{1}{n!} \nabla_\mu b_{\rho_1 \dots \rho_n}(x)\, y^{\rho_1} \dots y^{\rho_n} \tag{3}$$

which is determined by the relation

$$\exp\{-ib(x, y)\}(\partial^x_\mu - \partial^y_\mu) \exp\{ib(x, y)\} = i\omega_\mu(x, y) \tag{4}$$

and hence satisfies the generalized Mauerer-Cartan equation:

$$(\partial^x_\mu - \partial^y_\mu)\, \omega_\rho(x, y) - (\partial^x_\rho - \partial^y_\rho)\, \omega_\mu(x, y) + i[\omega_\mu(x, y), \omega_\rho(x, y)] = 0. \tag{5}$$

Under the transformations (2) $\omega_\mu(x, y)$ is transformed as the Yang-Mills field $b_\mu(x)$.

The infinite set of covariant differential conditions for the inverse Higgs effect,[6,8] which express the Goldstonians $b^i_{\mu_1 \dots \mu_n}(x)$ for $n \geqslant 2$ in terms of $b_\mu(x)$ and its derivatives, is now represented by a single equation

$$y^\mu[\omega_\mu(x, y) + b_\mu(x)] = 0 \tag{6}$$

or, taking into account Eq. (4):

$$y^{\mu}(\partial_{\mu}^{x}-\partial_{\mu}^{y})\exp\{-ib(x,y)\}=iy^{\mu}b_{\mu}(x)\exp\{-ib(x,y)\}. \tag{7}$$

Its solution is a contour functional of the field $b_{\mu}(x)$ along a rectilinear path from the point $x+y$ to the point x:

$$\exp\{-i\bar{b}(x,y)\}=T\exp\{i\int_{x+y}^{x}b_{\mu}(\xi)d\xi^{\mu}\}=T\exp\{-i\int_{0}^{1}y^{\mu}b_{\mu}[x+(1-\beta)y]d\beta\}, \tag{8}$$

where T denotes ordering in the matrices Q^{i} in the interval $0\leqslant\beta\leqslant 1$. The Cartan equation (3), written in terms of the minimum Goldstonian $\bar{b}(x,y)$, takes the form

$$\bar{\omega}_{\mu}(x,y)=-b_{\mu}(x)+\frac{1}{2}G_{\mu\rho}(x)y^{\rho}+\sum_{n\geqslant 2}\frac{1}{(n+1)!}\nabla_{\rho_1}\cdots$$
$$\cdots\nabla_{\rho_{n-1}}G_{\mu\rho_n}y^{\rho_1}\cdots y^{\rho_n}, \tag{9}$$

where $G_{\mu\rho}=\partial_{\mu}b_{\rho}-\partial_{\rho}b_{\mu}-i[b_{\mu},b_{\rho}]$ is the standard Yang-Mills rotor and ∇_{ρ} is the covariant derivative for the associated G_0 group representation.

Thus, the "stringed" functional of the gauge fields, which has been discussed in depth in the literature,[1,2,9-11] in our treatment is the most economical representation for the cosets G/G_0. We should emphasize that, in contrast to the standard approach, we determine the covariants by differentiating its finite points, which corresponds to an infinitesimal rotation of the entire path as a whole around the point $x+y$ [Eq. (4)], rather than by varying the separate parts of the path.

In its usual formulation, the inverse Higgs effect selects a straight path in the multiplicity of paths between the points $x+y$, x. However, without violating the transformation law (2) we can choose as a typical G/G_0 coset a functional along any curved path [such that it can be expanded in a series in $b(x,y)$ with respect to y_{μ}]. This corresponds to equating the symmetrical parts of the covariant derivatives of the Goldstonians, beginning with $\nabla_{\mu}b_{\rho_1\rho_2}(x)$, to some combinations of the covariant derivatives of the rotor $G_{\rho\mu}(x)$, rather than to zero, like in Eq. (6). Any such element can always be represented in the form:

$$\exp\{i\tilde{b}(x,y)\}=\exp\{i\bar{b}(x,y)\}\exp\{i\,\tilde{\tilde{b}}(x,y)\},\ (\tilde{\tilde{b}}'(x,y)=e^{-i\lambda(x)}\tilde{\tilde{b}}(x,y)e^{i\lambda(x)}), \tag{10}$$

where the nonminimal factor $\exp\{i\tilde{\tilde{b}}(x,y)\}$, which describes the deviation from the linear path, is expressed in terms of the degrees of the covariant derivatives of $G_{\rho\mu}(x)$.

3. The main relation (4) has a form which is characteristic of expansions that determine the covariant derivatives in nonlinear σ-models for the principal chiral fields. Therefore, the Yang-Mills theory may be interpreted as a sector of a nonlinear σ-model for the main bilocal chiral field $b(x,y)$ in the group G_0, which is distinguished by the conditions (5) and (6); moreover, by definition, $b_{\mu}(x)=\partial_{\mu}^{y}b(x,y)|_{y=0}$ and $b(x,0)=0$. It is interesting to determine whether the Yang-Mills equations can be

represented as a certain differential covariant condition for the form $\omega_\mu(x, y)$, which is supplementary relative to the "kinematic" conditions (5) and (6). From the point of view of the hypothesis of total integrabilitiy of the Yang-Mills theory, it is desirable that this condition should be of first order in the derivatives.

In the Abelian case, the equations of motion for the field $b_\mu(x)$ (without the sources) are equivalent to the condition for "conservation" of $\bar{\omega}_\mu(x, y)$ in y:

$$\partial_\mu^y \bar{\omega}^\mu(x, y) = 0. \tag{11}$$

In the non-Abelian case, this equivalence is valid to an accuracy to the third order in y and it cannot be raised to higher orders without using the terms with higher derivatives.[1)] We emphasize that for the self-dual fields ($G_{\mu\rho} = \pm 1/2\epsilon_{\mu\rho\lambda\nu}G^{\lambda\nu}$) and light-similar sections $y^2 = 0$ in Eq. (8) the condition (11) is satisfied for each order in y. It is unclear whether the self-duality $b_\mu(x)$ follows from Eq. (11) at $y^2 = 0$.

It is interesting to note that in the general case $\omega_\mu(x, y)$ can be transverse in y in the solutions of the Yang-Mills equations, if we reject the linear condition (6) and define the G/G_0 classes by the functionals (10) corresponding to the curved paths. The condition (11) reduces to an equation for the functional $\tilde{\tilde{b}}(x, y)$, which is solvable for each order in y. It is tempting to assume that there is a correlation between the classes for solutions for the Yang-Mills equations and the integration path in (10).

4. We showed that the Yang-Mills theory in its standard, nonsymmetric phase allows a natural embedding in the bilocal, nonlinear σ model in the G_0 group. This leads us to assume that the symmetric phase of the theory, which is related to the gauge-invariant vacuum, should be described in terms of the corresponding bilocal linear σ model. In the simplest case $G_0 = SU(2)$ the easiest way to linearize the basic transformation law (2) involves examination of the bilocal matrix $U(x, y) = U^\circ(x, y) + i\tau^k U^k(x, y)$, which is transformed according to the law (2), but it does not satisfy the exponentiation condition $UU^\dagger = I$. The infinite set of fields in its expansion in y is transformed by the action of the K group linearly and uniformly. A more detailed discussion will be given in a separate article.

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[1)]Note a possible similarity to the recent results of Witten *et al.*[3]

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