

# New class of field theories

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It is shown that the field theories, which are equivalent to classical gases with generalized charges having certain discrete symmetries, are integrable.

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In statistical physics an important role is played by two-dimensional lattice models of magnets, which in the classical case are described by the Hamiltonian

$$H^{(n)} = \pm \frac{1}{2} \sum J_{ij} \mathbf{s}_i \cdot \mathbf{s}_j, \quad (1)$$

where  $\mathbf{s}_i$  are unit vectors specified in the lattice points, which can take on values in the entire  $n$ -dimensional sphere [continuous symmetry group  $O(n)$ ]<sup>(1)</sup> or at certain points (discrete symmetry groups).<sup>(2)</sup> The sign  $\pm$  corresponds to the antiferro- (AF) and ferro- (F) magnets, respectively (for diminishing  $J_{ij}$ ).

The quadratic form of Eq. (1) in  $\mathbf{s}_i$  allow us to express the statistical sum  $Z^{(n)}$  in terms of the variables coupled with  $\mathbf{s}_i$ . Using the new variables  $Z^{(n)}$  has the form of a generating functional of the field theory for the lattice. For example, for the Ising AF model ( $n = 1$ )

$$Z^{(1)} = \sum_{\{s_i\}} \exp(-\beta H^{(1)}) = \int D\phi_i \exp \left\{ -\frac{1}{2} \sum \phi_i J_{ij}^{-1} \phi_j + \sum V(\sqrt{\beta} \phi_i) \right\},$$

where  $V(x) = \ln \cos x$ . (2)

Conversion to the Ising F model corresponds to the substitution  $\sqrt{\beta} \rightarrow \pm i\sqrt{\beta}$ , i.e., Laplace transformation replaces Fourier transformation. As a result,  $Z^{(1)}$  remains the same, except  $V(x) = \ln \operatorname{ch} x$ .<sup>(3)</sup>

On the other hand, it is known that the field theories with  $V(x) = \cos x$  and  $V(x) = \operatorname{ch} x$  are integrable,<sup>(4)</sup> and the first one is equivalent to the Coulomb gas<sup>(5)</sup> [transformation for gases is similar to the transformation (2),<sup>(5,6)</sup>] whose Hamiltonian has the form of Eq. (1) with  $n = 1$ . Therefore, in going over to a gas with the Hamiltonians from (1) with  $J_{ij} = -1/2\pi \ln |\mathbf{x}_i - \mathbf{x}_j|$  (the spins  $\mathbf{s}_i$  play the role of generalized charges), we obtain local field theories with  $V$  that do not contain the logarithm.<sup>(7)</sup> As a result, we obtain the field theories with the following potentials for the gases with the

Hamiltonian  $H^{(n)}(n \geq 2)$ :

$$V_n(x) = (x/2)^{\frac{2-n}{2}} I_{\frac{n-2}{2}}(x) \quad \text{for } F \text{ gas,} \quad (3a)$$

$$V_n(x) = (x/2)^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(x) \quad \text{for } AF \text{ gas.} \quad (3b)$$

where  $x = \sqrt{\beta} |\vec{\phi}|$ , and  $|\vec{\phi}| = [\sum_1^n \phi_i^2]^{1/2}$ . We note that for the  $F$  gas with  $n = 0$  (Newton gas) we obtain the Liouville field theory.<sup>[8]</sup>

A question arises concerning the properties of these theories. Some of them for the theories with  $V_n$  from Eq. (3b) are given in Ref. 7. It is not clear at present whether these theories are integrable, because at  $n > 1$  their symmetry is continuous. It would be desirable, therefore, to examine first the field theories that are equivalent to the gases with discrete groups.

We shall begin with the  $F$  gases. The potential  $V$  in the general case has the following form

$$V(\vec{\phi}) = \sum_{i=1}^N \exp(\mathbf{s}^{(i)} \cdot \vec{\phi}), \quad \sum_{i=1}^N \mathbf{s}^{(i)} = 0, \quad \vec{\phi} = \sqrt{\beta} \vec{\phi}, \quad (4)$$

where  $\sum_{i=1}^N$  is the sum over all positions of the spin  $\mathbf{s}^{(i)}$  in the sphere. We write the potentials for certain simple, nontrivial groups.

1)  $n = 2$ ,  $Z_n$  symmetry groups.

$$N = 3, \quad V(\vec{\phi}) = e^{\phi'_2} + 2e^{-\frac{1}{2}\phi'_2} \text{ch} \frac{\sqrt{3}}{2} \phi'_1,$$

$$N = 4, \quad V(\vec{\phi}) = 2[\text{ch} \phi'_1 + \text{ch} \phi'_2].$$

2)  $n = 3$ , symmetry groups—regular polyhedron groups. We shall label them according to the number of vertices  $N$ .

$N = 4$ , tetrahedron group, see general equation (5).

$N = 6$ , octahedron group,  $V(\vec{\phi}) = 2\sum_1^3 \text{ch} \phi'_i$ .

3)  $n$  arbitrary. The order of the first nontrivial group is  $n+1$ , to which corresponds  $V$  from Eq. (4), where

$$\mathbf{s}^{(i)} = (a_n, b_n a_{n-1}, b_n b_{n-1} a_{n-2}, \dots, b_n \dots b_i, 0, \dots), \quad i = 1, 2, \dots, n, \quad (5)$$

$$\mathbf{s}^{(n+1)} = (1, 0, \dots), \quad a_n = -1/n, \quad b_n^2 = 1 - a_n^2.$$

In writing  $V$  for groups of  $(n+1)$  order, we always used a coordinate system of the following type. The  $e_n$  axis is directed along  $\mathbf{s}^{(n+1)}$ . After projection on the  $(n-1)$  dimension space, we obtain a  $(n-2)$  dimensional sphere with radius  $b_n$ . This sphere contains vectors corresponding to the projections of the remaining  $n$  vectors of the original figure. The same procedure is used for them. The next group has the order  $2n$ ,

and  $V(\vec{\phi}'_i) = 2\sum_1^n \text{ch } \phi'_i$  corresponds to it. In this case the  $s^{(i)}$  vectors coincide with the coordinate axes. We note that the lattice models with interaction of the nearest neighbors, which have the aforementioned discrete symmetries of the  $(n+1)$ -type groups, are self-dual. This is a consequence of the presence of only one excitation energy.<sup>12,9)</sup>

It follows from  $V$  for the  $2n$ -order group that these field theories are integrable. It is very likely that the field theories corresponding to the  $(n+1)$ -order groups are also integrable (due to their distinction). We shall demonstrate this for  $n+1=3$  ( $Z_3$  group) by reduction to the "two-dimensional Toda chain." The latter is a "two-dimensional" generalization of the regular Toda chain,<sup>10)</sup> which is also integrable.<sup>11)</sup> Its Lagrangian is

$$L_T = \frac{1}{2} (\partial_\mu x_i)^2 - V_T(x_{i+1} - x_i), \quad V_T(x_i) = \sum_1^M \exp(x_i), \quad x_{i+M} = x_i. \quad (6)$$

Let  $M=3$ . In the normal coordinates<sup>12)</sup> Eq. (6) can be rewritten as follows:

$$L_T = \frac{1}{2} (\partial_\mu q_i)^2 - V(q_i), \quad V(q_i) = e^{-\sqrt{2}q_2} + 2 \text{ch} \left( \sqrt{\frac{3}{2}} q_1 \right) e^{\frac{\sqrt{2}}{2} q_2}. \quad (7)$$

If we assume that the "center of gravity" is  $q_3 = (1/\sqrt{3})(\sum_1^3 x_i) = 0$ ,  $-\sqrt{2}q_2 = \phi'_2$ , and  $\sqrt{2}q_1 = \phi'_1$ , then Eq. (7) becomes the Lagrangian of the field theory corresponding to the  $Z_3$  group with a doubled coupling constant. Unfortunately, the field theory for the groups with  $n+1 > 3$  could not be converted into the Toda chain with  $M=n+1$ . The possibility of their integrability is being studied. The two-dimensional Toda chain with arbitrary  $M$  is equivalent to a certain "gas" field theory. Specifically,  $M=4$  corresponds to the "isotropic"-type potential of the Ashkin-Teller model, which is also self-dual in the lattice. It is conceivable that each Toda chain corresponds to a self-dual model in the lattice.

We shall discuss briefly the theories corresponding to the AF gases with discrete groups. The real  $V$  are obtained only for the groups with an inversion center. The simpler, nontrivial groups have the order  $2n$  with the vectors lying on the axes. They correspond to the integrable potentials  $V = 2 \sum_1^n \cos \phi'_i$ . The question concerning the integrability of the remaining  $V$  is still open.

At the same time, the properties of the AF gases with a continuous symmetry<sup>17)</sup> stated in Ref. 7 (with a slight modification for the groups without the inversion center) are transferable to the AF gases with a discrete symmetry, since they can have  $s^{(i)}$  vectors of the same length, and the gas, as a whole, is neutral.

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<sup>1</sup>H. E. Stanley, Phys. Rev. Lett. 20, 589 (1968).

<sup>2</sup>R. B. Potts, Proc. Camb. Phil. Soc. 48, 106 (1952).

<sup>3</sup>G. Baker, Phys. Rev. 126, 2071 (1962); M. Kac and E. Helfand, J. Math. Phys. 4, 1078 (1963).

- <sup>4</sup>V. E. Zakharov, D. D. Fadeev, and L. A. Takhtadzhyan, Dok. Akad. Nauk SSSR **219**, 1334 (1974) [Sov. Phys. Dokl. **19**, 824 (1975)]; M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. **30**, 1262 (1973).
- <sup>5</sup>S. Coleman, Phys. Rev. **D11**, 2088 (1975); J. Frolich, Comm. Math. Phys. **487**, 233 (1976).
- <sup>6</sup>A. M. Polyakov, Zh. Eksp. Teor. Fiz. **55**, 1026 (1968) [Sov. Phys. JETP **28**, 533 (1969)]; Nucl. Phys. **B120**, 429 (1977).
- <sup>7</sup>S. A. Bulgadaev, Phys. Lett. (in press).
- <sup>8</sup>V. A. Andreev, TMF **29**, 213 (1976).
- <sup>9</sup>L. Mittag and M. J. Stephen, J. Math. Phys. **12**, 441 (1971); V. S. Dotsenko, Zh. Eksp. Teor. Fiz. **75**, 1083 (1978) [Sov. Phys. JETP **48**, 546 (1978)].
- <sup>10</sup>M. Toda, Progr. Theor. Phys. Suppl. **45**, 174 (1970); S.V. Manakov, Zh. Eksp. Teor. Fiz. **67**, 543 (1974) [Sov. Phys. JETP, **40**, 269 (1975)]; H. Flaschka, Progr. Theor. Phys. **51**, 703 (1974).
- <sup>11</sup>A. V. Mikhaïlov, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 443 (1979) [JETP. Lett. **30**, 414 (1979)].
- <sup>12</sup>G. L. Kotkin and V. G. Serbo, Sb. Zadach po Klassicheskoi Mekhanike (Coll. of Problems on Classical Mechanics), M., 1977.