Quasiclassical spectrum of an anisotropic Landau-Lifshits ferromagnet

Yu. N. Kafiev
Institute of Mathematics, USSR Academy of Sciences, Siberian Division

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The quasiclassical excitation spectrum is calculated, using the periodic one-soliton solutions of the Landau-Lifshits equation.

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Among several completely integrable equations recently found and investigated, an extremely interesting one from the viewpoint of physical applications is the Heisenberg nonlinear "spin string" (HNSS)—the continuous analog of the corresponding spin chain—described by the equation:

$$S^a_t = \epsilon^{abc} S^b S^c_{xx}; \qquad S^a S^a = 1. \tag{1}$$

An inverse, scattering method solution has been developed[1] for Eq. (1), and its reduction to a nonlinear Schrödinger equation has been carried out,[2] apparently indicating its complete integrability. The authors of Ref. 3, using their method of integration with respect to the spin variables, obtained the quasiclassical spectrum for the HNSS and found that it is identical to Bethe's exact quantum answer.[4]

Recently, Borovik[5] aroused interest in the Landau-Lifshits nonlinear spin string (LLNSS), taking anisotropic effects into consideration:

$$S^a_t = \epsilon^{abc} S^b S^c_{xx} + \beta S^3 \epsilon^{3ab} S^b; \qquad S^a S^a = 1 \tag{2}$$

(in this paper we consider only $\beta > 0$). Borovik indicated the L-A pair for Eq. (2) and found the special case of a one-soliton solution. In this paper we will calculate the quasiclassical LLNSS spectrum, using the periodic solution of Eq. (2). It should be noted that the Borovik solution is not the most general since it does not take account of the Doppler variation of the spin precession frequency for $v \neq 0$.

Introducing spherical coordinates, we write:

$$S^a = (\cos\phi \sin\theta,\ \sin\phi \sin\theta,\ \cos\theta) .$$

The system (2) is reduced to two equations:

$$(\cos\theta)_t = (\sin^2\theta \phi_x)_x , \tag{3a}$$

$$\sin\theta\phi_t = \beta\cos\theta\sin\theta + \theta_{xx} - \phi_x^2\sin\theta\cos\theta. \tag{3b}$$

For the one-soliton solution $[\theta = \theta\,(x - vt)]$ Eqs. (3) are easily integrated and the general solution with correct boundary conditions is:

$$\phi = \phi_o + \left(\epsilon^2 + \frac{v^2}{4}\right)t + v\int^x \frac{dx}{1 + \cos\theta(x - vt)},$$

$$\cos\theta = 1 - \frac{2(\epsilon^2 + \beta)}{\sqrt{\left(\epsilon^2 + \frac{v^2}{4}\right)^2 + \beta v^2}\left\{\mathrm{ch}^2\left[(x - vt)(\epsilon^2 + \beta)^{1/2}\right] - \frac{1}{2}\right\} + \frac{v^2}{8} + \frac{\epsilon^2}{2} + \beta}. \tag{4}$$

A solution in the form (4) was previously obtained in Ref. 6. Let us point out that because of (3a) the quantity

$$Q_3 = \int_{-\infty}^{\infty} (1 - \cos\theta)\,dx \tag{5}$$

does not depend on time and its forms, together with the Hamiltonian H, the set of LLNSS integrals of the motion. Jevicky and Papanicoulaou[3] showed that for the HNSS the result of a quasiclassical quantization can be interpreted as the replacement Q_3 by the integer m, while preserving the classical relationship between energy and momentum $E = 8[1 - \cos(P/2)]/Q_3$. The authors of Ref. 6 assumed that for $\beta \neq 0$ one can also set $Q_3 = m$ and can relate E to P by classical expressions.

We will calculate explicitly the quasiclassical spectrum for the LLNSS and show that the assumption[6] is valid only as $\beta \to 0$. In the general case Q_3 depends on m in a very complicated manner and it is impossible to ascribe to it the meaning it has in the isotropic model, i.e., the number of magnons.

Following the method of Ref. 7, quantum field theories define as quasiclassical the computation of the functional integrals near periodic solutions of the field equations, and the spectrum is found from the poles of the Green's function

$$G(E) = i\,\Sigma\int_0^{\infty} dT e^{iET}\, 2\pi i \Delta_1 e^{iS_{cl}}. \tag{6}$$

We have ignored the quantum corrections in (6); they do not exist at $\beta = 0$.[3] In our case they can lead to a renormalization of β and are unimportant to us. We will impose a periodicity condition on the one-soliton solution (4)

$$\frac{2\pi}{\epsilon^2} = \frac{T}{l} \equiv \tau; \qquad l = 1, 2\ldots. \tag{7}$$

Because of space limitations in this paper we ignore the motion of the center of mass, i.e., we assume $v=0$ everywhere. For the physical quantities we then obtain [here $\tilde{Q}_3=(\sqrt{\beta}/4)Q_3$]

$$E=\int_{-\infty}^{\infty} H\,dx=\frac{8}{\sqrt{\beta}}\epsilon^2\tilde{Q}_3-4(\epsilon^2+\beta)^{1/2},$$

$$P=\int_{-\infty}^{\infty}\frac{S^1S^2_x-S^2S^1_x}{1+S^3}\,dx=2\pi,$$

$$\tilde{Q}_3=\frac{1}{2}\ln\left[\frac{\epsilon^2+2\beta+2\sqrt{\beta(\beta+\epsilon^2)}}{\epsilon^2}\right]. \tag{8}$$

Eliminating ϵ^2, we find the classical relationship between E and $\tilde{Q}_3$

$$E=4\sqrt{\beta}\left[\frac{2\tilde{Q}_3}{\mathrm{sh}^2\tilde{Q}_3}-\mathrm{cth}\tilde{Q}_3\right]. \tag{9}$$

Using the spin Lagrangian of Ref. 3, we find the action S_{cl}

$$S_{cl}=\int_0^T dt\int_{-\infty}^{\infty} dx\left[\frac{S^1_tS^2-S^1S^2_t}{1+S^3}-H\right]$$

$$=-T\left\{\frac{2\epsilon^2}{\sqrt{\beta}}\ln\left[\frac{\epsilon^2+2\beta+2\sqrt{\beta(\beta+\epsilon^2)}}{\epsilon^2}\right]+4(\epsilon^2+\beta)^{1/2}\right\}. \tag{10}$$

The quantity Δ_1 in Eq. (6) corresponds to the freedom in the choice of the initial conditions (Φ_0 in (4)) and is equal to $\Delta_1=(1/l^{1/2})\,2\tau^{1/4}/(2\pi+\beta\tau)^{1/4}$. After substitution of Δ_1 and (10) into (6) and taking account of the periodicity conditions (7), we compute the integral by the stationary phase method. A stationary point with respect to τ lies at $2\pi/\tau_0=E^2/16-\beta$; in this case we obtain for $G(E)$ the geometrical series

$$G(E)=\mathrm{const}\sum_{l=1}^{\infty}\exp(-ilw)f(E)=\frac{e^{-iw}}{1-e^{-iw}}f(E),$$

where

$$w=\frac{4\pi}{\sqrt{\beta}}\ln\left[\frac{E+4\sqrt{\beta}}{E-4\sqrt{\beta}}\right],$$

the form of $f(E)$ is unimportant to us. Poles occur in $G(E)$ for $w = 2\pi m$; hence we find

$$E_m = 4\sqrt{\beta}\,\mathrm{cth}\,\frac{m\sqrt{\beta}}{4} \cdot \tag{11}$$

As $\beta \to 0$, the spectrum (11) is identical to the exact quantum answer at $P = 4\pi$ for the HNSS,[3] $E_m = 16/m$ and corresponds to the quantization of Q_3. However, for β different from zero it is seen from (9), (11) that Q_3 is bounded above and depends on m in a complicated manner. It is obvious that the effect of anisotropy is not trivial and for $m\sqrt{\beta}/4 \sim 1$ the spectra are radically different: $E_m(\beta)$ ceases to depend on m. Unfortunately, it is impossible to compare the spectrum (11) with the exact quantum answer. It is easy to establish that the LLNSS is not the limit of any chain; the exact quantum spectrum for it is unknown. The difference of the spectra, as well as the symmetry properties for the Heisenberg and Landau-Lifshits "spin strings," has an adverse effect on the question[5] of the possibility of reducing one equation to the other. Taking account of the center of mass motion is of considerable interest for the LLNSS; since its equations, like the nonlinear Schrödinger equation and HNSS, are not Galilean invariant, the dependence on momentum will be nontrivial. This question will be considered in a separate paper.

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