Violent instability of one-dimensional forceless magnetic field in a rarefied plasma

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The equilibrium distribution function, increment and criterion of violent instability in collisionless and weakly-colliding regimes are obtained for a periodic one-dimensional forceless magnetic field.

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Violent instability (Thirring-mode) of a one-dimensional forceless magnetic field has been studied in detail in a hydrodynamic approximation both numerically [1] and analytically. [2] However, the hydrodynamic approximation is unjustified for many applications with rarified plasma (the solar corona, magnetosphere, certain laboratory experiments) and the plasma must be considered collisionless or weakly-collisional. [3,4] Below we shall obtain for these conditions the criterion and instability increment of a one-dimensional forceless magnetic field of the form

$$\mathbf{B}_0 = B_0 \{ \cos \alpha z, \sin \alpha z, 0 \}. \tag{1}$$

This field constitutes an exact solution of the magnetic hydrostatic equations for a negligibly small value of the plasma parameter $\beta = 8\pi p/B^2$:

$$\mathbf{B} \times \mathrm{rot}\, \mathbf{B} = 0, \ \mathrm{div}\, \mathbf{B} = 0. \tag{2}$$

In a broader sense, the field in Eq. (1) may be considered as a local approximation of any field with a small β and given extent α.

The equilibrium distribution function is defined in terms of the integrals of motion in the field (Eq. (1)):

$$f_a = f_a\left(v_x^2 + v_y^2 + v_z^2 + \frac{2p_a}{m_a}\phi,\ v_x + \frac{p_a}{m_a c} A_x,\ v_y + \frac{p_a}{m_a c} A_y\right), \tag{3}$$

where α denotes the type of particles ($\alpha = e, i$), m_α is mass, e_α is electron and ion charge, **A** and ϕ are vector and scalar potentials determined from Maxwell's equations in which the charge and current densities are expressed in terms of the distribution function (Eq. (3)). In particular, we shall examine for a field in Eq. (1) and $\phi = 0$ the following distribution function

$$f_a = \frac{n_0}{(2\pi m_a)^{3/2} T_{a\parallel} T_{a\perp}^{1/2}} \exp\left\{-\left[\frac{m_a}{2T_{a\perp}} v_z^2 + \frac{m_a}{2T_{a\parallel}}\left\{\left(v_x + \frac{p_a B_0}{c\, m_a}\frac{\Delta T_a}{T_{a\perp}}\frac{\cos az}{a}\right)^2\right.\right.\right.$$

$$\left.\left.\left. + \left(v_y + \frac{e_a B_0}{c m_a}\frac{\Delta T_a}{T_{a\perp}}\frac{\sin az}{a}\right)^2\right\}\right]\right\}, \tag{4}$$

where the field step is uniquely related to the difference ΔT_α of the longitudinal $T_{\alpha\parallel}$ and transverse $T_{\alpha\perp}$ temperatures:

$$a = \left(\frac{4\pi e^2 n_0}{c^2} \sum_a \frac{\Delta T_a}{m_a T_{a\perp}}\right)^{1/2}. \tag{5}$$

The existence of a resonant surface $\mathbf{k}\cdot\mathbf{B} = 0$ is necessary for violent instability.[3,5] It should be pointed out that in reality the condition for the occurrence of strong currents and, in particular, the Thirring instability is integral and it requires closure of the resonant magnetic lines of force.[6,7] Implicitly, this is understood also for the condition $\mathbf{k}\cdot\mathbf{B} = 0$ which may be written in the form $\mathbf{B}\cdot\mathrm{grad}\boldsymbol{\Phi} = 0$ where $\boldsymbol{\Phi}$ is an arbitrary function, for example a space charge potential. Evidently, we may require that $\mathbf{k}\cdot\mathbf{B} = 0$ only for closed magnetic lines. In this respect the case of the one-dimensional forceless field [Eq. (1)] is degenerate: for any $\mathbf{k}$ of the form $\mathbf{k} = \{k_x, k_y, 0\}$ there exists a surface $\mathbf{k}\cdot\mathbf{B} = 0$ and, one would think, the resonance surfaces continuously fill the space.[8] However, in reality the magnetic field may not be forceless everywhere (the Shafranov theorem).[9] Therefore, if we consider Eq. (1) simply as an approximation to the field $\mathbf{B}$ for a limited region, only those magnetic lines of force will be resonant (unique, to use the terminology adopted in Ref. 6) which close on themselves outside this region. In particular, such lines may not occur at all and, in such a case, there will be neither violent instability nor, all the more, resonance "overlapping."[8]

If, however, unique lines actually exist in the volume under observation, investigation of violent instability is reduced to the solution of a problem in two regions: external and internal near the resonant surface $\mathbf{k}\cdot\mathbf{B} = 0$.

We shall consider perturbations of the following form:

$$F_1(x, y, z, t) = F_1(z) \exp\{i(k_x x + k_y y) + \omega t\}. \tag{6}$$

In the external region where there is no dependence on the collision frequency ν_c the problem is reduced to consideration of adiabatically slow perturbations within the framework of ideal MHD. A solution of the problem in the internal region for a collisionless ($\omega \gg \nu_c$) and weakly-collisional ($\omega < \nu_c$) plasma was obtained in Ref. 4. The step of a logarithmic derivative function $\psi = B_{12}/B_0$ is respectively:

$$\Delta' = \frac{2\sqrt{\pi}\,\kappa_0^2\,\omega}{\omega\,\bar{v}_e\, a} \quad (\text{at } \omega >> \nu_c), \tag{7}$$

where v_e is the thermal speed of electrons, $\kappa_0 = k_0/\alpha$, $k_0 = \omega_{pe}/c, \omega_{pe}$ is the plasma frequency, $\kappa = k/\alpha$ or

$$\Delta' = \frac{8\pi^{1/4}\Gamma(11/4)}{3} \frac{\kappa_0^2 \omega^{3/2}}{\kappa v_e a \nu_c^{1/2}} \quad (\text{at} \quad \omega < \nu_c). \tag{8}$$

In analogy with Ref. 2, matching the solutions in the external and internal regions, we get the following equation:

$$\psi'' + \psi\{(1-\kappa^2) - \Sigma \Delta' \delta(kB_0)\} = 0, \tag{9}$$

in which the presence of δ-functions takes into account possible discontinuities of the derivative near the surfaces $\mathbf{k}\cdot\mathbf{B}_0 = 0$. We should note that Eq. (3) has the form of the Schrödinger equation for a particle moving in a periodic potential.[10] The solution of this equation yields the dispersion equation for unstable harmonics, which relates ω, κ and the "longitudinal" wave number Q that characterizes the displacement of perturbations at the neighboring resonant surfaces:

$$\cos Q\pi = \cos\sqrt{1-\kappa^2}\,\pi + \Delta' \frac{\sin\sqrt{1-\kappa^2}\,\pi}{2\sqrt{1-\kappa^2}}. \tag{10}$$

Hence, instability in both collisionless and weakly-collisional regimes will occur only if

$$\kappa^2 + Q^2 < 1. \tag{11}$$

The instability increment of a collisionless plasma is

$$\omega = \frac{(\cos Q\pi - \cos\sqrt{1-\kappa^2}\,\pi)\,\kappa\sqrt{1-\kappa^2}}{\sqrt{\pi}\,\kappa_0^2 \sin\sqrt{1-\kappa^2}\,\pi} v_e a, \tag{12}$$

and weakly-collisional

$$\omega = \left\{\frac{3}{4\pi^{1/4}\Gamma(11/4)} \frac{(\cos Q\pi - \cos\sqrt{1-\kappa^2}\,\pi)\,\kappa\sqrt{1-\kappa^2}}{\kappa_0^2 \sin\sqrt{1-\kappa^2}\,\pi} v_e a \nu_c^{1/2}\right\}^{2/3}. \tag{13}$$

The maximum increment is attained for $Q = 0$ and at $\kappa \sim \kappa_0^{1/2}$ in the case of a collisionless plasma or $\kappa \sim \kappa_0^{1/4} (\nu_c/v_e\alpha)^{1,2}$ in the case of a weakly-collisional plasma, when Eqs. (7) and (8) are still applicable. The maximum instability increment for both collisionless and weakly-collisional plasma is

$$\omega_{max} \backsim \kappa_0^{-3/2} (v_e a). \tag{14}$$

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