New effects in the Sine-Gordon model for a large coupling constant

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It is shown that for $\gamma \to 8\pi$ a rich spectrum of particles appears in the Sine-Gordon model, and the soliton scattering matrix is a discontinuous function of the coupling.

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The sine-Gordon (SG) model

$$L = \frac{1}{\gamma} \int dx \left\{ \frac{1}{2} \left(\partial_\mu u \right)^2 + M_0^2 (\cos u - 1) \right\}, \quad \frac{8\pi}{\gamma'} \equiv \frac{8\pi}{\gamma} - 1, \quad \mu \equiv \pi - \frac{\gamma}{8} \tag{1}$$

is equivalent to the massive Thirring model (MTM):[1,2]

$$L = \int dx \left\{ i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi - \frac{1}{2} g \left(\bar{\psi} \gamma_\mu \psi \right)^2 \right\} . \tag{2}$$

The MTM mass spectrum was obtained in Ref. 3 and 4, and the S-matrix in Ref. 5. This work shows that for $16\pi/3 < \gamma < 8\pi$ the mass spectrum and the S-matrix are given by completely new equations. Direct calculations are carried out in the MTM

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but, following tradition, the language of the SG model is used. Exact quantum solutions can be found in SG using the quantum method for the inverse problem.[6,7] Equations used in this work are a direct matrix generalization of the equations obtained in Ref. 8.

It has been shown that the observables in the MTM model may be conveniently calculated using the eigenfunctions of the TM quantum Hamiltonian:[10]

$$\phi = \int d^l x\, \chi^{a_1 \dots a_l}(x_1, \dots, x_l \mid \beta_1, \dots, \beta_l)\, \psi^{+a_1}(x_1) \dots \psi^{+a_l}(x_l) \mid 0 > ,$$

$$\chi^{a_1 \dots a_l}(x_1, \dots, x_l \mid \beta_1, \dots, \beta_l) = \prod_{k=1}^{l} \chi^{a_k}(x_k \mid \beta_k) \prod_{j > i} \times \exp\left\{ \frac{i}{2} \epsilon(x_j - x_i) \Phi(\beta_j - \beta_i) \right\}, \tag{3}$$

$$\chi^a(x \mid \beta) = \begin{pmatrix} \exp\left\{ \frac{1}{2} \beta \right\} \\ \exp\left\{ -\frac{1}{2} \beta \right\} \end{pmatrix} \exp\{ i x m \,\mathrm{sh}\, \beta \}, \qquad \psi^a(x) \mid 0 > = 0 .$$

Here $\epsilon(x)$ is a sign function, β is the pseudoparticle speed, and $\exp\{i\Phi(\beta)\}$ is the scattering matrix of two pseudoparticles:

$$\Phi(\beta) = -i \ln\left\{ -e^{-2i\mu}(e^{\beta} - e^{2i\mu})(e^{\beta} - e^{-2i\mu})^{-1} \right\} \quad \mu = \frac{\pi + g}{2} \, , \tag{4}$$

It appears that n-pseudoparticles form a bound state if the sign of

$$\sin(p\mu) \sin((n - p)\mu) \ , \quad p = 1, 2, \dots, n - 1 \tag{5}$$

is independent of p. In order to analyze the bound states, it is convenient to divide the entire interval $0 < \mu < \pi/3$ into segments:

$$\pi/(q + 2) < \mu < \pi/(q + 1) \, , \quad q = 2, 3, \dots \tag{6}$$

Clearly, each segment contains bound states with

$$n = 1, 2, \dots, q \, . \tag{7}$$

The masses of these states are negative, $m_n = m \sin(n\mu)/\sin\mu$. As is known, the coupling constant for the interactions (1) and (2) depends on the renormalization scheme.[1,2,4] Quantization by means of Eq. (3) yields coincidence of the mass spectra, if $g = \pi - \gamma/4$.

Most important is the question concerning the physical vacuum, i.e., the state with the lowest energy. It has been shown[9] that for $0<\gamma<16\pi/3$ the vacuum is a vector in the Fock space, in which all the elementary pseudoparticle states are filled with negative energy. For $16\pi/3<\gamma$ this is no longer true. The introduction of the bound state for two pseudoparticles in such a vacuum lowers the energy. In this paper it is shown that the physical vacuum differs in all the segments (6). In segment (6) the vacuum is made up of the pseudoparticle bound state with $n = 1,...,q$ (7). Moreover, all the states of these complexes are filled. Let us now proceed with calculations. For the sake of regularization let the system be placed in a periodic box of length L and construct an ultraviolet cut-off $|\mathrm{Re}\,\beta|<\Lambda$. Let β^n_j denote the allowed values of the real part of the bound state velocity for n-pseudoparticles in a vacuum. The periodicity equations for the vacuum wave function (3) were conventionally solved in the limit $L\to\infty$.[9,10] The stability requirement for β^n_j as $\Lambda\to\infty$ leads to the following renormalization equation (it agrees with the usual renormalization for $0<\gamma<16\pi/3$[8]):

$$m = (\pi - 2\mu)\, M \exp\left\{ \frac{\pi - 2\mu}{2\mu}\Lambda \right\}. \tag{8}$$

Let us consider excitation above the vacuum with zero charge. We shall make a hole in the nth condensate component with velocity β_n. Moreover, the permitted values of the vacuum particle velocities change. Let them be $\tilde{\beta}^n_j$. As $L\to\infty$, the periodicity conditions for the corresponding function (3) are converted to equations for $F_a(\beta^n_j/n) = (\beta^a_j - \tilde{\beta}^a_j)/(\beta^a_{j+1} - \beta^a_j)$:[9,11]

$$\Phi^{\bullet n}_a(\beta) = 2\pi f^\bullet_a(\beta/n) + \sum_{b=1}^{q} \int_{-\infty}^{\infty} \Phi^{\bullet a}_b(\beta - a)\, f^\bullet_b(a/n)\, da .$$

Here Eq. (4) is

$$f_a(\beta/n) = F_a(\beta + \beta_n/n),\quad \Phi^n_a(\beta)$$
$$= \sum_{j=0}^{n-1} \sum_{p=0}^{a-1} \Phi(\beta + i\mu(a - n + 2j - 2p)).$$

Using Fourier transformation we find that the zero components of the solution have the form

$$f^\bullet_l(\beta|l+1) = f^\bullet_{l+1}(\beta|l)$$

$$= \frac{1}{2\pi i}\frac{d}{d\beta}\ln\left\{ \frac{\exp\{\pi\beta/2\mu\} - i}{\exp\{\pi\beta/2\mu\} + i} \right\},\quad l = 1,\dots,q-1,$$

$$f^\bullet_q(\beta|q) = \frac{i}{2\pi}\frac{d}{d\beta}\int_0^\infty \left(\frac{dx}{x}\right) \frac{\mathrm{sh}(4\pi x/\gamma^\bullet_c - x/2)\,\mathrm{sh}(4ix\pi\beta/\mu\gamma^\bullet_c)}{\mathrm{sh}(x/2)\,\mathrm{ch}(4\pi x/\gamma^\bullet_c)}, \tag{9}$$

$$\gamma^\bullet_c/8\pi = \pi/\mu - q .$$

The observed values for the energy and charge have the conventional form:

$$E_n = m \frac{\sin(n\mu)}{\sin\mu} \operatorname{ch}\beta_n - m \sum_{b=1}^{q} \frac{\sin(b\mu)}{\sin\mu} \int_{-\Lambda}^{\Lambda} \operatorname{ch}\beta F_b^{\bullet}(\beta|n)\, d\beta ,$$

$$Q_n = -n + \sum_{b=1}^{q} b \int_{-\Lambda}^{\Lambda} F_b^{\bullet}(\beta/n)\, d\beta.$$

Using Eqs. (8) and (9) we find that E_n, Q_n, and the masses M_n are:

$$E_n = M_n \operatorname{ch}\theta_n , \; M_n = 2M \sin(n\mu)/\operatorname{tg}\mu, \; Q_n = 0, \; n = 1, \dots, q-1,$$

$$E_q = M_q \operatorname{ch}\theta_q , \quad Q_q = -\pi/2(\pi - q\mu), \tag{10}$$

$$M_q = M\left\{ \sin(q-1)\mu/\sin\mu + \operatorname{tg}\frac{\pi}{2}\left(\frac{\pi}{\mu} - q - 1\right)\sin q\mu/\sin\mu \right\}.$$

Here θ is the observed velocity $\theta = \pi\beta/2\mu$. We shall consider other excitations. We introduce an elementary pseudoparticle into the vacuum, having a positive energy and rate β_p. Similar calculations indicate that

$$f_q^{\bullet}(\beta|p) = \frac{1}{2\pi i}\frac{d}{d\beta}\ln\frac{\operatorname{ch}\frac{4\pi}{\gamma_c^{\bullet}}\left(\frac{\pi\beta}{\mu} + i\pi\right)}{\operatorname{ch}\frac{4\pi}{\gamma_c^{\bullet}}\left(\frac{\pi\beta}{\mu} - i\pi\right)}, \; f_l^{\bullet}(\beta|p) = 0, \; l = 1, \dots, q-1,$$

$$Q_p = \pi/(\pi - \mu q), \; E_p = 0. \tag{11}$$

The introduction of the bound state $(q+1)$ of a pseudoparticle (5) leads to the following results

$$f_q^{\bullet}(\beta/q+1) = \frac{1}{2\pi i}\frac{d}{d\beta}\ln\frac{\operatorname{sh}\frac{4\pi}{\gamma_c^{\bullet}}\left(i\pi + \frac{\pi\beta}{\mu}\right)}{\operatorname{sh}\frac{4\pi}{\gamma_c^{\bullet}}\left(i\pi - \frac{\pi\beta}{\mu}\right)}, \; f_l^{\bullet}(\beta|q+1) = 0,$$

$$l = 1, \dots, q-1,$$

$$Q_{q+1} = Q_p = \pi/(\pi - \mu q), \; E_{q+1} = 0. \tag{12}$$

The introduction of any other bound state (5) into the vacuum leads to the results- $E_n = 0$, $Q_n > Q_p$. Therefore, all the excitations above the vacuum have been considered.

Evidently, in the sector with vacuum charge the energy of all the excitations is positive. Finally in segment (6), the theoretical spectrum contains a $(q-1)$ neutral particle and one charged one (10) which we shall call a soliton. The states for the soliton scattering by an antisoliton were constructed in a conventional way[3,9] These are two holes in the qth component of the condensate with velocities β_1 and β_2 and a bond, depending on the spatial parity of the state, this is either an elementary pseudoparticle or a $(q+1)$ bound state for a pseudoparticle with a velocity $(\beta_1+\beta_2)/2$.

Let us proceed with the calculation of the S-matrix. The observed hole scattering phase has the form $\ln S^a_b = 2\pi i f_a(\beta/b)$. The hole bond scattering phase has the form $\ln\ U^a_+(\beta) = 2\pi i f_a(\beta/2|q+1)$, $\ln\ U^a_-(\beta) = 2\pi i f_a(\beta/2|p)$. Using these equations it is easy to calculate the S-matrix for the physical particles (10). The following matrix elements of (9) are different from unity:

$$S^{\,l}_{l+1}(\theta) = \frac{ie^{\theta}+1}{e^{\theta}+i},\quad l = 1,\dots,q-1,\ \theta = \theta_{l+1}-\theta_l$$

and $S^q_q(\theta)$. Equations (9), (11), and (12) show that soliton-antisoliton scattering matrix is given by the Zamolodchikov equation[5,12] with the substitution $\gamma\to\gamma_c$:

$$S^{\pm}_{s\bar{s}}(\theta|\gamma) = U_{\pm}(\theta|\gamma_c)\,S(\theta|\gamma_c),\quad \gamma_c = 8\pi\,\frac{\pi-\mu q}{\pi-\mu(q-1)},$$

$$S(\theta|\gamma) = \exp\left\{-\int_0^{\infty}\frac{dx}{x}\,\frac{\mathrm{sh}(8i\theta x/\gamma')\,\mathrm{sh}\left(\frac{4\pi}{\gamma'}x-\frac{1}{2}x\right)}{\mathrm{sh}(x/2)\,\mathrm{ch}(4\pi x/\gamma')}\right\},$$

$$U_+(\theta|\gamma) = \frac{\mathrm{sh}\,\frac{4\pi}{\gamma'}(i\pi+\theta)}{\mathrm{sh}\,\frac{4\pi}{\gamma'}(i\pi-\theta)},\quad U_-(\theta|\gamma) = -\frac{\mathrm{ch}\,\frac{4\pi}{\gamma'}(\theta+i\pi)}{\mathrm{ch}\,\frac{4\pi}{\gamma'}(\theta-i\pi)}.$$

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