Lagrangian description of flute drift turbulence of θ-pinch plasma

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Two-fluid hydrodynamics is used to derive a system of equations suitable for the description of advanced drift fluxes of particles and the energy in a θ-pinch plasma. A general stability criterion is obtained for an inhomogeneous plasma in a straight field relative to drift oscillations of the flute type. The nonlinear regime is investigated qualitatively.

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We derived from two-fluid hydrodynamics a system of equations suitable for the description of advanced drift fluxes of particles and the energy in a θ-pinch plasma. These fluxes can lead to a cooling of the plasma within a time eBa^2/cT (a is the radius of the pinch). A general stability criterion is obtained for an inhomogeneous plasma in a straight field relative to drift oscillations of the flute type. The nonlinear regime is investigated qualitatively.

1. The anomalously large escape of heat in a direction transverse to the magnetic field from the hot region of a plasma that is in mechanical equilibrium may be due to oblique (drift) heat fluxes in the inhomogeneous plasma. The heat redistribution connected with these fluxes leads to the onset of relatively slow plasma motions with velocity $v_{dr} \sim cT/eBa$, much lower than the speed of sound. An important role is played in such processes by the generation of the magnetic field. The last-mentioned effect in a nonmagnetized plasma was considered in[1]. In some configurations, the drift fluxes turned out to be closed so no energy is carried out. It was shown in[2,3] that such configurations may turn out to be unstable to nonresonant buildup of drift oscillations of the flute type. Drift motions with scales exceeding the ionic gyroradius and with times exceeding the ion-ion collision time can be described with the aid of the equations of two-fluid hydrodynamics.[4] In the case $(\omega_H\tau)_{e,i} \gg 1$ and $(8\pi nT/B^2)(\omega_H\tau)_e \gg 1$ we can neglect in these equations the small terms that describe dissipative effects (the transverse thermal conductivity, the diffusion of the magnetic field, the thermoelectric power, and the Joule and viscous heats). A simplified system of equations describing oblique nondissipative heat flow and the plasma convection it causes has, as will be shown below, a special structure that makes it possible to separate the description of the heat fluxes from that of convection.

To simplify the exposition we consider the case $\beta = 8\pi nT/B^2 \gg 1$ (wall containment,[5,6] which is of particular physical interest, since it is fraught with the danger that generation can cause the magnetic field to decrease significantly in part of the pinch cross section, and the heat losses may increase sharply. As $\beta \to \infty$ we can put $\mathbf{v}_e = \mathbf{v}_i = \mathbf{v}$, since $|\mathbf{v}_e - \mathbf{v}_i| \ll |\mathbf{v}| \sim v_{dr}$. We start from the equations for the particle and energy balance (cf. [4]).

$$\frac{\partial n}{\partial t} + \mathbf{v}\nabla n + n\,\mathrm{div}\,\mathbf{v} = 0,$$

$$\frac{\partial p_e}{\partial t} + \mathbf{v}\nabla p_e + \frac{5}{3}p_e\,\mathrm{div}\,\mathbf{v} = -\frac{2}{3}\mathrm{div}\,\mathbf{q}_{e\wedge} = -\frac{5}{3}\mathrm{div}\frac{p_e}{B}\left[\nabla\frac{p_e}{n}\times\hat{z}\right], \tag{1}$$

$$\frac{\partial p_i}{\partial t} + \mathbf{v}\nabla p_i + \frac{5}{3}p_i\,\mathrm{div}\,\mathbf{v} = -\frac{2}{3}\mathrm{div}\,\mathbf{q}_{i\wedge} = -\frac{5}{3}\mathrm{div}\frac{p_i}{B}\left[\nabla\frac{p_i}{n}\times\hat{z}\right], \tag{2}$$

$$b \equiv eB/c\,. \tag{3}$$

It is assumed that $\mathbf{B}\|\hat{z}$, $\mathbf{v}\|\hat{z}$, and $(\hat{z}\cdot\nabla)\equiv 0$, where $\hat{z}$ is the unit vector in the z direction. In drift motion, the inertia does not play an important role and the equation of plasma motion can be replaced by the equations (at $nT \gg B^2/8\pi$):

$$p_e + p_i = p = \mathrm{const}\,, \tag{4}$$

$$\left(\mathrm{rot}\, n\frac{d\mathbf{v}}{dt}\right)_z = 0. \tag{5}$$

From $\partial\mathbf{B}/\partial t = -c\,\mathrm{curl}\,\mathbf{E}$ and from the electron-motion equation $-en\mathbf{E} - (en/c)\mathbf{v}\times\mathbf{B} - \nabla p_e = 0$, in which the small forces due to friction against the ions, electron inertia, and the viscous force have been omitted, we obtain an equation for $b = eB/c$:

$$\frac{\partial b}{\partial t} + \mathbf{v}\nabla b + b\,\mathrm{div}\,\mathbf{v} = \left(\mathrm{rot}\frac{\nabla p_e}{n}\right)_z = \left[\nabla\frac{1}{n}\times\nabla p_e\right]_z, \tag{6}$$

in which the right-hand side describes the generation of the magnetic field.

The system (1)—(6) conserves the total energy and entropy of the plasma. It makes it possible to ascertain in detail the conditions and the rate of development of the drift fluxes that lead to loss of heat to the walls.

Adding (2) and (3) we obtain, taking (4) into account,

$$\mathrm{div}\,\mathbf{v} = \frac{1}{p}\left\{\left[\nabla\frac{p_i}{b}\times\nabla\frac{p_i}{n}\right]_z - \left[\nabla\frac{p_e}{b}\times\nabla\frac{p_e}{n}\right]_z\right\}. \tag{7}$$

Substituting this expression for div v in (1) and (6) and in the equation for $q = p_e - p_i$ obtained by subtracting (3) from (2), we obtain

$$\frac{dn}{dt} = \frac{\partial n}{dt} + \mathbf{v}\nabla n = \frac{1}{2b^2}[\nabla q\times\nabla b]_z + \frac{q}{nb^2}[\nabla b\times\nabla n]_z + \frac{1}{2bn}[\nabla n\times\nabla q]_z, \tag{8}$$

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \mathbf{v}\nabla q = -\frac{5}{6}\frac{p^2 - q^2}{b^2 n^2}[\nabla b\times\nabla n], \tag{9}$$

$$\frac{db}{dt} = \frac{\partial b}{\partial t} + \mathbf{v}\nabla b = \frac{q}{bn^2}[\nabla b\times\nabla n] + \frac{1}{2nb}[\nabla q\times\nabla b]. \tag{10}$$

Owing to the pecularities of the right-hand sides of Eqs. (8)—(10), the system

(5), (7), (8)—(10) can be simplified by changing from the Euler coordinates x and y to Lagrangian coordinates x_0 and y_0. In this case

$$\frac{d}{dt}\Big|_{\text{Euler}} = \frac{\partial}{\partial t}\Big|_{\text{Lagrange}}$$

$$[\nabla u\times\nabla v]_{\text{Euler}} \equiv \frac{D(u,v)}{D(x,y)} = \frac{D(u,v)}{D(x_0,y_0)}\,\frac{D(x_0 y_0)}{D(x,y)} = \frac{n}{n_0}\,\frac{D(u,v)}{D(x_0,y_0)} = \frac{n}{n_0}[\nabla u\times\nabla v]_{\text{Lagrange}}$$

for arbitrary functions u and v. Here $D(u,v)/D(x,y)$ is the Jacobian of the transformation $(x,y)(u,v)$, $n=n(x_0,y_0,t)$ is the running density and $n_0=n(x_0,y_0,0)$ the initial density in the specified Lagrangian point. Equations (8)—(10) take in Lagrangian coordinates the form

$$\frac{\partial n}{\partial t} = \frac{n}{n_0}\left\{\frac{1}{2b^2}[\nabla q\times\nabla b]_z + \frac{q}{nb^2}[\nabla b\times\nabla n]_z + \frac{1}{2nb}[\nabla n\times\nabla q]_z\right\}, \tag{11}$$

$$\frac{\partial q}{\partial t} = \frac{n}{n_0}\left\{-\frac{5}{6}\,\frac{p^2-q^2}{b^2n^2}[\nabla b\times\nabla n]_z\right\}, \tag{12}$$

$$\frac{\partial b}{\partial t} = \frac{n}{n_0}\left\{\frac{q}{bn^2}[\nabla b\times\nabla n]_z + \frac{1}{2nb}[\nabla q\times\nabla b]_z\right\}. \tag{13}$$

The system (11)—(13) is autonomous; its solution yields the dependence of n, q, b on x_0, y_0, t and then, using (5) and (7), we can change back to the Euler coordinates. Many properties of the drift turbulence, however, can be investigated on the basis of Eqs. (11)—(13) in Lagrangian form. Thus, Eq. (13) can be written in the form $\partial b/\partial t + \mathbf{w}.\nabla b = 0$, from which follows the important conclusion that $B_{\min}$ and $B_{\max}$ do not change, and no demagnetization can occur in the θ pinch. Furthermore, any state in which the level lines n, q, and b coincide (in particular, axially-symmetric states) is stationary for Eqs. (11)—(13). In the investigation of the stability of these states, the equations for the perturbations $\tilde{n}$, $\tilde{q}$, and $\tilde{b}$ contain derivatives of $\tilde{n}$, $\tilde{q}$, and $\tilde{b}$ along the level lines of the stationary states. It follows therefore that the eigenfunctions are localized on these level lines, and the stability criterion can be expressed in terms of the local values of the quantities n, q, and b and their derivatives.

An analog of the system (11)—(13) can be obtained also for arbitrary β. We present the stability criterion and the growth rate for the general case

$$\Gamma = \frac{c\,(T_e + T_i)\,\gamma^{1/2}}{2\,eB\,(\gamma + 2\beta^{-1})}\sqrt{-\,Xk} \tag{14}$$

$$X > 0 \quad \text{for stability}, \tag{15}$$

where k is the wave vector of the perturbation,

$$X = \kappa_B^2[(\gamma^2 - 4\beta^{-2}) + \theta^2(\gamma - 1)(4\beta^{-2} - \gamma)] + \kappa_{(T_e - T_i)}^2[\gamma\theta^2]$$

$$+ \kappa_{(Te - Ti)}\kappa_B[4(\gamma - 1)\theta^2\beta^{-1}] + \kappa_n\kappa_B[-2\gamma(\gamma + 2\beta^{-1}) + 2\theta^2(\gamma - 1)(\gamma + 2\beta^{-1})],$$

$$\vec{\kappa}_A \equiv \nabla \ln A (A = B, n, T_e - T_i); \quad \theta^2 = \left(\frac{T_e - T_i}{T_e + T_i}\right)^2; \quad \gamma = 5/3 . \tag{16}$$

It is seen from (16) that at comparable T_e and T_i the terms containing θ^2 can be neglected ($\theta^2 < 1/9$ at $0.5 < T_e/T_i < 2$), and the stability criterion takes the simpler form

$$\vec{\kappa}_B^2(\gamma - 2\beta^{-1}) - 2\gamma\vec{\kappa}_n\vec{\kappa}_B > 0 \text{ for stability.} \tag{17}$$

It follows from (17) that the states obtained by rapid heating (with an electron beam, liner, or laser) of a homogeneous plasms in a uniform magnetic field, and having $\kappa_n \approx \kappa_B$ as a result of the freezing-in, are unstable.

To illustrate the nonlinear regime, we consider the case of a strong temperature deviation $q^2(x_0, y_0, t = 0) = p^2$. It follows from (12) that this equality is preserved at all times, and it follows from (11) and (13) that

$$\frac{\partial}{\partial t}\left(\frac{n}{b}\right) = 0, \text{i.e.}, \frac{n}{b} = f(x_0, y_0).$$

Substituting this equation in (11), we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{n}\right) + \frac{q}{n_0(x_0, y_0)}\frac{1}{n}\left[\nabla f \times \nabla \frac{1}{n}\right] = 0.$$

This equation is analogous to the equation $\mathring{v} + vv' = 0$; "toppling" of the profiles of n and b takes place after a finite time, and discontinuities of these quantities appear. An important role near the discontinuities is played by dissipative effects, which determine the structure of the discontinuity. The description of drift fluxes on the basis of Eqs. (11)—(13), which do not contain small parameters, seems to us to be quite informative and at the same time convenient for investigation by numerical methods.

[1]L. A. Bol'shov, Yu. A. Dreĭzin, and A. M. Dykhne, Pis'ma Zh. Eksp. Teor. Fiz. **19**, 288 (1974) [Sov. Phys. JETP **19**, 168 (1974)].

[2]B. B. Kadomtsev, Zh. Eksp. Teor. Fiz. **37**, 1096 (1959) [Sov. Phys. JETP **10**, 780 (1960)].

[3]L. V. Mikhaĭlovskaya, Zh. Tekh. Fiz. **37**, 1974 (1967) [Sov. Phys. Tech. Phys. **12**, 1451 (1968)].

[4]S. I. Braginskiĭ, Voprosy teorii plazmy (Problems of Plasma Theory), Vol. 1, M. 1962.

[5]G. I. Budker, Proc. Sixth European Conf. on Plasma Physics and Controlled Fusion **2**, 136, Moscow, 1973.

[6]G. E. Vekshteĭn, D. D. Ryutov, M. D. Spektor, and P. Z. Chebotaev, Zh. Prikl. Mekh. Tekh. Fiz. **6**, 3 (1974).