Self-similar solutions with a diverging and a collapsing shock wave in general relativity theory

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Self-similar solutions with converging and collapsing spherical shock waves in an ultrarelativistic gas are obtained in general relativity theory.

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In general relativity theory, in analogy with classical gas dynamics, it is natural to seek solutions with shock waves in the class of self-similar spherically symmetrical solutions. These solutions can be represented in the conformally-static form

$$ds^2 = \exp(2\tau)(\sigma\exp(\nu(r))d\tau^2 - \sigma\exp(\lambda(r))dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where $\sigma = \pm 1$. In fact, the mapping $R = \exp(\tau + \zeta)$, $t = \exp(\tau + \psi(\zeta))$, where $\zeta = \ln r$ and $d\psi/d\zeta = \exp(\lambda - \nu + 2\zeta)$, transforms the metric (1) into a metric of self-similar form[1]:

$$ds^2 = \exp(\nu_0(t/R))dt^2 - \exp(\lambda_0(t/R))dR^2 - R^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2}$$

This paper deals with the solutions of Einstein equations of type (1) with a hydrodynamic material energy-momentum tensor T_i^j and with an equation of state $p = k\epsilon$, $0 < k < 1$. The energy density ϵ and the 4-velocity of the matter u are given by

$$\epsilon = \bar{\epsilon}(r)\exp(-2\tau),$$

$$(u^0, u^1, u^2, u^3) = \exp(-\tau)\left(\left(\frac{\sigma\exp(-\nu)}{1-u^2}\right)^{1/2}, \text{and} \left(\frac{\sigma\exp(-\lambda)}{1-u^2}\right)^{1/2}, 0, 0\right),$$

where $u = v^\sigma$, $\sigma = \pm 1$, and v is the three-dimensional radial velocity of matter.

The system of Einstein's equations

$$R_0^0 - R_1^1 = T_0^0 - T_1^1,\quad R_2^2 - \tfrac{1}{2}R = T_2^2,\quad T_{1;k}^k = 0 \tag{3}$$

reduces, after eliminating ϵ from the equation $R_{01} = T_{01}$, to a closed system of three ordinary differential equations in the variable $\zeta = \ln r$ on the functions $Q(\zeta) = \exp(\zeta + (\lambda - \nu)/2)$, $w(\zeta) = d\nu/d\zeta$, $u(\zeta)$. The equation $R_1^1 - \frac{1}{2}R = T_1^1$ is satisfied by virtue of the system (3).

An important feature of the obtained system is the presence of surfaces $V_\pm$: $u = \pm k^{1/2}$ of non-continuability of the solutions, on which the derivative $du/d\zeta$ reverses sign passing through infinity (everywhere on $V_\pm$, with the exception of line I, see below). This non-continuability of the solutions means that a shock wave appears in the real solution of the problem. The conditions on the discontinuity[2] lead to the following rule for joining together the solutions: the functions ν, λ, r, w, and Q are continuous on the discontinuity, the values of the velocity u and of the energy density ϵ on the two sides of the discontinuity (1 and 2) are connected by the relations $u_1u_2 = k$ and $\epsilon_1/\epsilon_2 = (u_2^2 - k)/k(1-u_2^2)$. The only solutions with physical meaning are those in which: (a) the shock wave is a compression wave, (b) the solution is defined for all $r > 0$.

An investigation of the system (3) shows that conditions (a) and (b) are satisfied by trajectories of the following three types. I. Separatrices emerging from the unstable singular point $Z_1(q = Q/u = -3(1+k)/(1+3k),\ u = w = 0)$. II. Separatrices emerging (with increasing ζ) from the segment I_1 of the unstable singular points on line I: $u = -k^{1/2}$ and $w(1-k) + 8Qk^{3/2}(1+k)^{-1} = 4k$ on the surface V_-. III. Separatrices that enter in the segment I_1 and the segment I_2 of the attracting (at $|u| < k^{1/2}$) singular point on the line I. We list the properties of the corresponding solutions.

I. The separatrices of the singular point Z_1 correspond to solutions with a diverging shock wave. In the reference frame (2) these solutions are continued to the symmetry center $R = 0$. After passage of the shock wave, the metric has no singularity at the center, and the gas velocity v and the energy density at $R/t \ll 1$ have asymptotic limits $v \approx (\frac{2}{3}(1+k))R/t$ and $\epsilon \approx c/t^2$. Solutions exist in which the coordinate R has along the trajectories of the gas motion, after the passage of the shock wave, an arbitrary number of oscillations. In the region ahead of the shock wave, the solutions are continued in the static reference frame at $t_1 \geq 0$. The shock wave radius is $R_0 = C_0t_1$, $C_0 > 0$. The formation of the shock wave (explosion) takes place at the symmetry center $R_1 = 0$ at $t_1 = 0$. At this instant the gas velocity is directed towards the center, the energy density is $\epsilon = C_1/R^2{}_1$, the metric is flat at infinity, and at the symmetry center the metric has a conical singularity.

II. The separatrices emerging from the segment I_1 and continued smoothly through the surface V_- also correspond to solutions with a diverging shock wave. The behavior of these solutions behind the shock wave, however, differs substantially from the solutions of type I. In particular, in the solutions of type II the space-like sections at $t \neq 0$ are products of a two-dimensional sphere by a straight line (as in the well-known Kruskal solution), and have an "orifice" that is contracted to a point at the instant when the shock wave emerges from the center $R = t = 0$.

III. The separatrices that enter in the segment $I_1 + I_2$ and are smoothly continued through the surface V_- correspond to solutions with a collapsing shock wave. In terms of the coordinates (2) the radius of the shock waves is $R_0 = -C_0t$ $(t < 0,\ C_0 > 0)$. The spacelike sections at $t \neq 0$ are topologically products of a two-dimensional sphere by a straight line and have an "orifice" that is contracted to a point upon collapse of the shock wave at the center $R = t = 0$.

The foregoing brief description of the obtained solution indicates that these solution have a number of properties in common with the known solutions of the explosion problem[3] and of the problem concerning the converging shock wave[4-6] in classical dynamics, as well as that the solutions of these problems acquire essentially new properties in the general theory of relativity.

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