New type of additive topological conservation laws for multidimensional solitons

L. É. Gendenshteĭn

Khar'kov Physicotechnical Institute, Ukrainian Academy of Sciences
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It is shown that to specify nontrivial boundary conditions that lead to the appearance of topological conservation laws for multidimensional solitons it is not necessary that the set of the fields that minimize the Hamiltonian be compact; it may be sufficient for it to contain compact intersecting sets. An example is presented of such a model, which is a certain generalization of the sine-Gordon model to include the multidimensional case. This gives rise to new additional topological conservation laws.

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It is known that soliton solutions can appear in field theories with degenerate vacuum.[1–4] An interesting feature of such solutions are distinctive conservation laws called topological.[2,3] These conservation laws are connected with the presence of nontrivial boundary conditions for solutions that have finite energy. We denote by Φ the set of values of the fields corresponding to the minimum of the potential energy, and by R the set corresponding to the boundary $r \to \infty$ (two points in the two-dimensional case and a sphere in the three-

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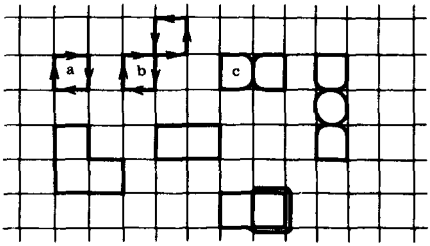

FIG. 1.

dimensional one). For the solution to have a finite energy it is necessary to have fields $\phi \in \Phi$ at infinity. We are thus dealing with mappings $R \rightarrow \Phi$. Field configurations that are topologically not equivalent (belonging to different homotopic classes) can not go over into one another because of the infinite potential barrier, and the topological characteristics of the mapping $R \rightarrow \Phi$ are thus conserved quantities.

We are interested in conservation laws of the type of the conservation law for the *number* of solitons. Such conservation laws become particularly interesting if it is assumed that the soliton solutions can correspond to particles,[1,2,5] while the conservation laws of such particles are regarded as topological.[2]

It is obvious that in the models of interest to us must, first, provide for n-soliton solutions for arbitrary integer n (positive and negative). Second, the topological characteristic corresponding to the number of particles (solitons) must be additive.

The first condition means that the number of homotopic classes must be infinite, and the second that the corresponding homotopy group must be Abelian.

For example, the one-dimensional Higgs model is not of this type because of the limited number of possible ways of specifying the boundary conditions. It is known that the conserved quantity in this model is only the parity of the number of solitons ("kinks").

The models that satisfy our requirements are, for example, the one-dimensional sine-Gordon model or the 't Hooft-Polyakov multidimensional solitons.[1,2]

For our purposes it is important to note that an infinite number of homotopic classes of the mapping $R \rightarrow \Phi$ is realized in one-dimensional ($D=1$) and multidimensional ($D>1$) models in entirely different ways.

In one-dimensional models it is due to the infinite number of discrete minima of $U(\phi)$, and the conserved quantity is $\phi(\infty) - \phi(-\infty)$. In multidimensional models the conserved additive quantity is the *degree of the mapping* $R \rightarrow \Phi$.[2-4]

It is customary to assume for multidimensional models (in contrast to the one-dimensional ones), however, that the limit of R must in this case be mapped on the *entire* set Φ, and in this case nontrivial mappings exist only when Φ is a compact set. In our opinion, this restriction is excessive and as a result

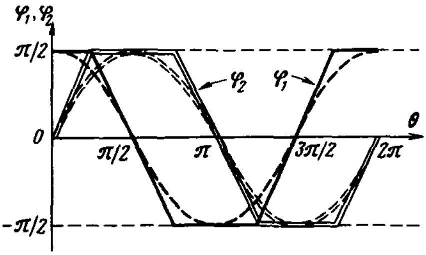

FIG. 2.

sight is lost of a large number of models that have new interesting properties. We consider first an actual example of one such model, and then note possible generalizations.

Let the potential-energy density in the Lagrangian be

$$U(\phi) = \beta \prod_{i=1}^{D} \cos^2\phi_i \,, \tag{1}$$

(we do not include terms with spatial derivatives of the fields). This model can be regarded, in a certain sense, as a multidimensional generalization of the sine-Gordon model, but we note that in our case *the number of fields is equal to the dimensionality D of the space*. $U(\phi)$ is minimal at $\phi_i = (k+\frac{1}{2})\pi$, where k are integers. We confine ourselves for the time being to $D=2$ for the sake of argument. In this case we can imagine for $U(\phi)$ a "quilted" surface, i.e., Φ is a set of point on a lattice in the field space and is not compact. What kind of mappings $R\to\Phi$ are specified in this case by nontrivial boundary conditions? The simplest of them is marked in Fig. 1 by the letter *a* and correspond to mapping of R on the boundary of one of the cells in field space. The corresponding stationary solution recalls qualitatively the solution for the Higgs potential. Indeed, for small ϕ the levels $U(\phi)=\text{const}$ are close to circles (spheres if $D=3$), and therefore the solution of the Euler equations

$$\Delta\phi_i = -2\beta \operatorname{tg}\phi_i\, U(\phi) \approx -2\beta\phi_i\, U(0) \tag{2}$$

under the boundary conditions defined above will be of the "hedgehog" type.[2] Moreover, a similar behavior will take place also at infinity: Fig. 2 shows the dependence of $\phi_{1,2}$ on the space angle θ as $r\to\infty$ for the two-dimensional case (in the Higgs model this would be sinusoids, also shifted by $\pi/2$—they are shown dashed). At any θ, we have in the asymptotic limit at least one $\phi=(k+\frac{1}{2})\pi$, i.e., $U(\infty)=0$, as is necessary for a localized solution. In addition, just as for the Higgs potential, we can specify an arbitrary degree of mapping on the surface (boundary) of the cell, determining by the same token the boundary conditions for the multisoliton (generally speaking, nonstationary) solutions.

The model in question, however, includes also other new types of nontrivial boundary condition. For example, Fig. 1(b) shows the mapping of R on a "figure-8" corresponding to one soliton and one "antisoliton," which, however, *cannot be annihilated*, since the figure-8 cannot be contracted to a point if the requirement $U(\infty)=0$ at each instant is to be preserved. Consideration of other types of mapping on Fig. 1 shows that new homotopic classes have appeared because Φ, while by itself not compact, it *contains compact intersecting subsets*. Let us describe the new types of mappings in the general case when Φ has such subsets.

We break up R into connected subsets R_i and denote by R_{ik} the intersection of R_i and $R_k (R_{ik} = R_i \cap R_k)$. We choose, next, intersecting subsets Φ^α from the entire set Φ, and put $\Phi^{\alpha\beta} = \Phi^\alpha \cap \Phi^\beta$. The sought mapping $R \to \Phi$ is specified as the aggregate of the continuous mappings

$$R_i \to \Phi^\alpha ; \qquad R_k \to \Phi^\beta \tag{3}$$

subject to the condition $R_{ik} \to \Phi^{\alpha\beta}$. Now each of the mappings (3) can have *its own* degree of mapping n_i^α, and thus, instead of one additive conserved quantity n (in the Higgs model) we have a set $\{n_i^\alpha\}$ (generally speaking, infinite) of such quantities.[1] If the Φ^α are different, then solitons with different masses and other characteristics can exist within the framework of one model.

We note in conclusion that, in considering nontrivial boundary conditions, we did not touch upon other conditions needed for the existence of localized nonsingular solutions (e.g., stability to scale transformations $\phi(x) \to \phi'(\lambda x)$. As a rule these questions have no bearing on the topological classification of the solutions, and various approaches to their solution are described, for example, in[3].

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[1]Different sets $\{n_i^\alpha\}$ are still not always sufficient for the determination of physically different solutions. For example, for the model with the potential (1) all the configurations $\{\Phi^\alpha\}$ that can be obtained from one another by translation or rotation (in field space) lead to physically equivalent solutions.

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