Multiple fractionation of wave structures in a nonlinear medium

N. A. Zharova, A. G. Litvak, T. A. Petrova, A. M. Sergeev, and A. D. Yunakovskiĭ
Institute of Applied Physics, Academy of Sciences of the USSR

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A new type of self-effect of wave fields in nonlinear media has been detected.

One of the standard equations of the theory of nonlinear waves is the hyperbolically parabolic nonlinear Schrödinger equation[1]

$$-iE_t + E_{xx} + E_{yy} - E_{zz} + E|E|^2 = 0. \tag{1}$$

This equation describes the self-effect of a broad class of waves, whose wave-vector surfaces have a saddle point (such waves are deep-water gravitational waves,[2,3] plasma oscillations in a magnetized plasma—upper hybrid oscillations,[4] cyclotron oscillations[5]). In contrast with the known three-dimensional scalar collapse, the nonlinear wave dynamics with such a dispersion law may lead to systematic production of wave structures of progressively smaller scale.

Let us consider the dynamics of a single wave packet $E(x,y,z)$ localized in space. From the standpoint of Eq. (1), the axially symmetric distributions collapse if the energy per unit length of the cell, $w = \int |E|^2 dxdy$, is higher than the critical energy and if the longitudinal modulation hinders the collapse. Using known integrals (1), we can show that the mean square of the longitudinal dimension of the wave packet,

$$\overline{b^2} = \int z^2 |E|^2 dxdydz / \int |E|^2 dxdydz$$

is a monotonically increasing function of time

$$(\overline{b^2})_{tt} = 2\int (4|E_z|^2 + |E|^4) dxdydz / \int |E|^2 dxdydz. \quad (2)$$

Equation (2) does not rule out, however, the occurrence of local singularities due to a transverse collapse at finite time intervals.

To resolve this problem, let us analyze the asymptotic stability of a two-dimensional, radially symmetric collapse with respect to spatial modulation along z.[1)] For this purpose, we switch from the laboratory frame to one that contracts uniformly toward a certain point ($R = 0, z = z_0$) (we choose $z_0 = 0$ in it). Transforming the field according to

$$E(x, y, z, t) = \frac{1}{a} u\left(\frac{R}{a}, \frac{z}{a}, \int \frac{dt}{a^2}\right) \exp\left(-\frac{ia_t R^2}{4a}\right), \quad (3)$$

we find from (1) the equation

$$\hat{L}u = -iu_\tau + \Delta_\xi u - u_{\eta\eta} + u|u|^2 = -u\frac{\xi^2}{4}\left(\frac{2a_\tau^2}{a^2}\tau - \frac{a_{\tau\tau}}{a}\right) - i\eta u_\eta \frac{a_\tau}{a}, \quad (4)$$

where $\xi = R/a$, $\eta = z/a$, $a = a(\tau(t))$, and $\int (dt/a^2)$ is a new time-measuring scale for which the time at which the singularity occurs corresponds to infinity. We know from the self-focusing theory[7] that a distribution uniform in z collapses in accordance with

$$a \sim \sqrt{\frac{t_0 - t}{-\ln(t_0 - t)}}, \quad (5)$$

so that the terms on the right side of (4) tend asymptotically $[\sim 1/|\ln(t_0 - t)|]$ to zero. Analysis based on the equation $\hat{L}u = 0$ shows that the principal ("Townes"[8]) mode of the transverse collapse, $u = V_0(\xi)\exp(-i\tau)$, is unstable with respect to amplitude perturbations, $V_0(\xi)$, of the type $V = A(\xi)\exp(\gamma\tau - i\kappa\eta)$. The maximum increment $(\mathrm{Re}\gamma)_{max} \simeq 0.75$ is reached at $\kappa_* \simeq 0.6$ and the function $\mathrm{Re}\gamma \sim \sqrt{\kappa}$, which was found in Ref. 9, is valid in the limiting case of the long-wave modulation ($\kappa \to 0$).

The data on the instability can be used to study the dynamics of structures extended in the z direction. If, on the other hand, self-compression involves a cell with the ratio of transverse and longitudinal dimensions $a_0/b_0 \gtrsim 0.1$ ($\kappa_0 \simeq 2\pi a_0/b_0 < \kappa_*$), then another mechanism, which prevents the appearance of a singularity, comes into play. We assume that the field-distribution focusing occurs independently in each cross section (in z). Replacement of variables of the type in (3)

$$E(x, y, z, t) = \frac{1}{a(z, t)} u\left(\frac{R}{a(z, t)}, \tau(z)\right) \exp\left(-\frac{i a_t R^2}{4a(z, t)}\right),$$

which is a generalized lens transformation[10] with the parameter a that depends on z, leads us to the equation

$$-iu_\tau + \mu u_\tau + \Delta_\xi u + u|u|^2 = 0, \tag{6}$$

where $\mu = a_0^2/2b_0^2$. Equation (6) is used for the cross section z_0, which corresponds to the field maximum $[E_z(z = z_0) = 0]$. In this equation the asymptotically vanishing

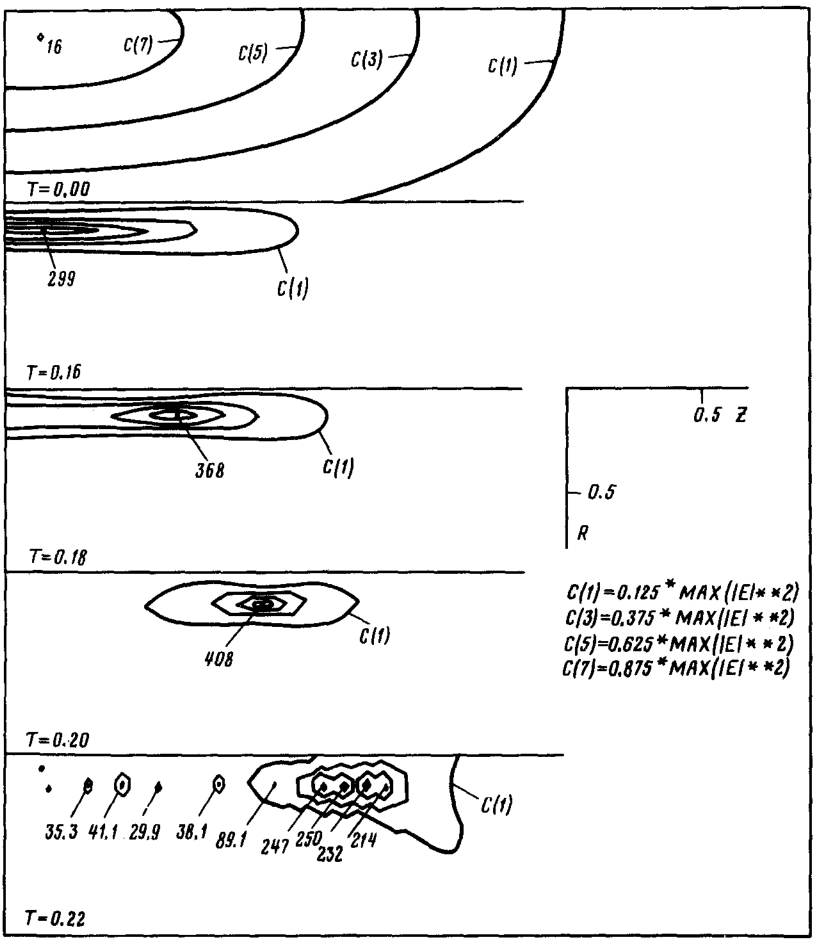

Fig. 1.

(in the limit $\tau \to \infty$) terms are ignored. The Townes mode in (6) decays at $\mu \neq 0$ due to the "outflow" of the field energy from the maximum in the z direction. We clearly see that this situation prevents compression of the overall distribution which is nonuniform in z and leads to the formation (in a manner similar to the two-dimensional case[11]) of two new small-scale regions (clusters) of highly concentrated field.

Evolution of the secondary structures follows a scenario similar to that described above. Transverse self-focusing in this case stems from the fact that the breakup of the initial distribution gives rise to a local increase of the field amplitude in the forming field clusters due to the redistribution of the oscillation energy density in the longitudinal direction. Consequently, in contrast with the three-dimensional collapse of oscillations in a medium with a positive dispersion, nonlinear field dynamics in our case typically includes a series of events involving transverse compression and longitudinal division of the three-dimensional structure, which accounts for the formation of progressively smaller-scale field clusters in the system.

The self-effect described here has been confirmed by numerical analyses of dynamically localized distributions in Eq. (1) for axial symmetry and wave packets with an initial Gaussian shape of the field,

$$E(r, z, 0) = A \exp\left(-\frac{R^2}{2a^2} - \frac{z^2}{2b^2}\right).$$

Figure 1 shows the lines for the function $|E(R,z,t)|^2$ and its local maxima at sequential instants of time for the parameters $A = 4$, $a = 1$, and $b = 4$. In the given version of the calculation we carefully analyze two stages of size reduction of the spatial field structure.

Determination of the total number of fractionation events is a complex problem. The evolution of a localized packet in an unbounded region of space apparently has a finite number of such events, since the average energy density per unit length decreases over time in the longitudinal direction [see Eq. (2)]. As numerical experiments have shown, the asymptotic solution (in the limit $t \to \infty$) is characterized in this case by a defocusing of the field in all directions.

The situation is quite different in a study of the self-effect in a system with periodic (along the z axis) boundary conditions. This situation corresponds to the initial excitation of an appreciable number of turbulence cells that interact between each other. Boundedness of the evolution space prevents the defocusing of the distribution as a whole and accounts for the endless recurrence of compression and division cycles at a progressively smaller scale.

[1)]Malkin[6] was first to analyze the stability of "supersonic" radially symmetric collapse of waves in a plasma.

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