Hidden supersymmetry of He^3-A

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The bound states of He^3-A, which correspond to the supersymmetry potential minimum produced by the texture of the orbital vector **1**, are "mirror degenerate": The particles with clockwise and counterclockwise spin, whose spectra are devoid of inversion symmetry, are in the zero-point-energy state.

Ho *et al.*[1] considered the effect of the orbital soliton on the Fermi-excitation spectrum in the A-phase of ^{3}He. They found that the spectrum of bound states with zero energy does not have mirror symmetry with respect to the plane perpendicular to the domain wall. A similar result was obtained by Combescot and Dombre[2] and Volovik.[3] In the present letter we analyze the Fermi-excitation spectrum in He^3-A in terms of supersymmetric quantum mechanics in the presence of texture in the orbital vector **1**.[4] We detected a "mirror degeneracy" (more on this point below) of zeroth-order modes.

The Gor'kov equation for a superfluid liquid with triplet pairing implies an eigenvalue problem $H\chi = E\chi$, where $\chi = \binom{\chi_1}{\chi_2}$ (see Refs. 1–3). In this case, the Hamiltonian H for He^3-A can be written in the form

$$H = \frac{\Delta_0}{p_F}\gamma_j\left(\mathbf{e}^j\cdot\hat{\mathbf{p}} - \frac{i}{2}\nabla\cdot\mathbf{e}^j\right) + \gamma_3\xi(\hat{\mathbf{p}}), \tag{1}$$

where $\vec{\Delta} = \mathbf{e}^1 + i\mathbf{e}^2$ is the orbital part of the order parameter (the spin variables are constant), $\vec{\gamma} = \{\sigma_1, -\sigma_2, \sigma_3\}$ (σ_j are the Pauli matrices), $\xi(\hat{\mathbf{p}}) = \hat{\mathbf{p}}^2/2m - \mu$, Δ_0 is a gap, p_F is the Fermi momentum, μ is the chemical potential, and $\hat{\mathbf{p}} = i\nabla$.

Let us consider the twist-texture of the orbital vector $\mathbf{1} = \mathbf{e}^1\times\mathbf{e}^2$:

$$\mathbf{l}(z) = l_x(z)\hat{\mathbf{x}} + l_y(z)\hat{\mathbf{y}}, \tag{2}$$

for which

$$\mathbf{e}^1 = -l_y(z)\hat{\mathbf{x}} + l_x(z)\hat{\mathbf{y}}, \quad \mathbf{e}^2 = \hat{\mathbf{z}}, \quad \nabla\cdot\mathbf{e}^j = 0. \tag{3}$$

Since $\mathbf{e}^j$ depend exclusively on the z coordinate, the momentum $\mathbf{p}_\perp = \{p_x, p_y\}$ is a motion constant and $\chi = \exp\{i(p_x x + p_y y)\}\Phi(z)$. In this case we can write the Hamiltonian H and the equation $H\chi = E\chi$ in the form

$$H = \frac{\Delta_0}{p_F}[-\sigma_2\hat{p}_z + \sigma_1 W(z)] + \sigma_3\xi(\hat{\mathbf{p}}), \tag{4}$$

$$(-\sigma_2\hat{p}_z + \sigma_1 W + \sigma_3\kappa - \epsilon)\Phi = 0. \tag{5}$$

Here

$$W = (\mathbf{l}\times\mathbf{p})_z, \quad \hat{p}_z = -id/dz, \quad \epsilon = (p_F/\Delta_0)E, \\ \kappa = \frac{p_F}{\Delta_0}\left(\frac{\hat{p}_z^2}{2m} + \xi_{\mathbf{p}_\perp}\right), \quad \xi_{\mathbf{p}_\perp} = \frac{\mathbf{p}_\perp^2}{2m} - \mu. \tag{6}$$

We assume that the scale dimension of the texture $L_z \gg p_F^{-1}$, so that $\kappa \simeq \kappa_{\mathbf{p}_\perp} \equiv (p_F/\Delta_0)\xi_{\mathbf{p}_\perp}$ and Eq. (5) becomes

$$\mathcal{H}^2\varphi \equiv \left[-\frac{d^2}{dz^2} + W^2 + \kappa_{\mathbf{p}_\perp}^2 + \sigma_3\frac{dW}{dz}\right]\varphi = \epsilon^2\varphi, \tag{7}$$

where H is a Hamiltonian corresponding to (5), with $\kappa = \kappa_{\mathbf{p}_\perp}$. The solutions of (5) are constructed from the solutions of (7): $\Phi = (\mathcal{H} + \epsilon)\varphi$. The Dirac equation, written in the form of a second-order equation for the vector potential $\mathbf{A}(z) = \{A_x(z), 0, 0\}$ reduces to (7), and $\mathbf{E} = 0$ (Ref. 5). Equation (7) was solved in Ref. 5 for certain forms of the potential, $\mathbf{A}(z)$ (see also Ref. 6).

For the states on the Fermi surface ($\xi_{\mathbf{p}_\perp} = 0$) Eq. (7) is a supersymmetric quantum-mechanics equation. This equation was analyzed in Ref. 4. However, in our case $W' = \text{curl}\,\mathbf{l}\cdot\mathbf{p} = \partial_z l_x p_y$, so that (7) has an additional parameter, p_y, which can change sign. We assume that W has one zero. The potential W^2 will then have the only supersymmetric minimum with zero-point energy. A very simple situation corresponds to the choice $W = p_x + p_y qz (q > 0)$, in which the Hamiltonian $H_0 = \hat{\mathbf{p}}_z^2 + \lambda^2(z - z_0)^2 + \lambda\sigma_3$, where $\lambda \equiv W' = p_y q$, and $z_0 = -p_x/\lambda$, corresponds to Eq. (7). The energy of zero-point oscillations of the oscillator is equal to $|\lambda|$ and the eigenvalues of the operator $\lambda\sigma_3$ are equal to $\pm|\lambda|$. If, for example, $p_y \geqslant 0$, the minimum of the ground-state energy will be $|\lambda| - \lambda = 0$. A change in the sign of p_y corresponds to the use of an eigenvalue $\lambda = -|\lambda|$ for the operator $\lambda\sigma_3$, i.e., a spin flip, and to a replacement of the coordinate of the oscillator center, z_0, by $-z_0$.

The state with zero-point energy in He^3-A (the ground state for H_0) thus has a distinctive "mirror degeneracy": for a particular sign of p_x (e.g., $p_x \geqslant 0$) the particles with a clockwise spin ($p_y \geqslant 0$) are situated on the negative z semiaxis and the particles with a counterclockwise spin ($p_y < 0$) are situated on the positive z semiaxis. Working from the symmetry considerations, we assume that the total contribution to the zero-mode current is zero. The current itself, however, must be studied separately.

To complete the description, we will find the spectrum and the eigenfunctions of (5) for $W = \lambda(z - z_0)$. In this case, Eq. (5) is completely analogous to the dirac equation for an electron in a constant magnetic field (see Refs. 5 and 7). As was already pointed out, however, there is an important exception which has to do with the doubling of the states of the given problem because of its dependence on the sign of λ. As a result, we can write the eigenvalues in Eq. (5) in the form

$$\epsilon_n = \pm [\kappa^2_{\mathbf{p}_\perp} + |\lambda|(2n + 1 + \mathrm{sgn}(\lambda)s)]^{1/2} \tag{8}$$

(where $s = \pm 1$ are the eigenvalues of the operator σ_3), and the eigenfunctions $\Phi^{\mathrm{sgn}}_{s,n}(\zeta)(\zeta = z - z_0)$ are

$$\Phi^+_{1,n-1} = \frac{1}{G_+}\begin{pmatrix}(\epsilon_n + \kappa_{\mathbf{p}_\perp})\psi_{n-1}\\ \sqrt{2|\lambda|n}\,\psi_n\end{pmatrix}, \quad \Phi^+_{-1,n} = \frac{1}{G_-}\begin{pmatrix}\sqrt{2|\lambda|n}\,\psi_{n-1}\\ (\epsilon_n - \kappa_{\mathbf{p}_\perp})\psi_n\end{pmatrix} \tag{9}$$

$$\Phi^-_{1,n-1} = \frac{1}{G_-}\begin{pmatrix}-\sqrt{2|\lambda|n}\,\psi_n\\ (\epsilon_n - \kappa_{\mathbf{p}_\perp})\psi_{n-1}\end{pmatrix}, \quad \Phi^-_{-1,n} = \frac{1}{G_+}\begin{pmatrix}(\epsilon_n + \kappa_{\mathbf{p}_\perp})\psi_n\\ -\sqrt{2|\lambda|n}\,\psi_{n-1}\end{pmatrix}, \tag{10}$$

where $\psi_n(\zeta)$ are the eigenfunctions of the oscillator ($\psi_{-1} = 0$), $G_\pm = [2\epsilon_n(\epsilon_n \pm \kappa_{\mathbf{p}_\perp})]^{1/2}$. The state with $n = 0$ is therefore doubly degenerate, while the remaining levels are quadruply degenerate (for each of the two signs of ϵ_n).

The state with $n = 0$, for which

$$\Phi^-_{-1,0} = \begin{pmatrix}\psi_0\\ 0\end{pmatrix}, \qquad \Phi^+_{-1,0} = \begin{pmatrix}0\\ \psi_0\end{pmatrix}, \tag{11}$$

has a curious feature. The spectrum $E_0 = |\xi_{\mathbf{p}_\perp}|(E_0 = -|\xi_{\mathbf{p}_\perp}|)$ is asymmetric: $\xi_{\mathbf{p}_\perp} > 0$ and $\xi_{\mathbf{p}_\perp} < 0$ ($\xi_{\mathbf{p}_\perp} < 0$ and $\xi_{\mathbf{p}_\perp} > 0$) for $\Phi^-_{-1,0}$ and $\Phi^+_{-1,0}$, respectively (in contrast with the results of Refs. 1–3). For $\xi_{\mathbf{p}_\perp} = 0$ this feature of the spectrum was noted above.

We note in conclusion that supersymmetry in Eq. (7) is broken for $\kappa_{\mathbf{p}_\perp} \neq 0$. We can introduce in this case the Hermitian charges,

$$Q_1 = [\sigma_1\hat{p}_z + \sigma_2(W + \kappa_{\mathbf{p}_\perp})]/\sqrt{2}$$

and

$$Q_2 = [\sigma_2\hat{p}_z - \sigma_1(W - \kappa_{\mathbf{p}_\perp})]/\sqrt{2},$$

which satisfy the algebra

$$Q_1^2 + Q_2^2 = H_\kappa, \qquad \{Q_1, Q_2\} = 2\kappa_{\mathbf{p}_\perp}\hat{p}_z, \tag{12}$$

where $H_\kappa = \hat{p}_z^2 + W^2 + \kappa_{\mathbf{p}_\perp}^2 + \sigma_3 W'$ is a Hamiltonian which corresponds to Eq. (7). If $\kappa_{\mathbf{p}_\perp} = 0$ [the following substitution $\epsilon^2 - \kappa_{\mathbf{p}_\perp}^2 \to \epsilon^2$ can be made in (7)], algebra (12) becomes supersymmetric quantum-mechanics algebra: $\{Q_i, Q_j\} = \delta_{ij} H_0$, $[Q_i, H_0] = 0$, and the number of states with zero-point energy is equal to the number of zeros in the function $W(z)$ (see Ref. 4).

[1]T. L. Ho, J. R. Fulco, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **52**, 1524 (1984); **54**, 1462 (1985).
[2]R. Combescot and Y. Dombre, Phys. Rev. Lett. **54**, 1461 (1985); Phys. Rev. B **33**, 79 (1986).
[3]G. E. Volovik, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 294 (1985) [JETP Lett. **42**, 363 (1985)].
[4]E. Witten, Nucl. Phys. **B185**, 513 (1981).
[5]G. N. Stanciu, Phys. Lett. **23**, 232 (1966).
[6]R. Gamonal, Phys. Rev. D **32**, 2846 (1985).
[7]M. H. Johnson and B. A. Lippman, Phys. Rev. **76**, 828 (1949).

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