Hamiltonian systems with even and odd Poisson brackets; duality of their conservation laws

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The existence of Hamiltonian systems with even and odd Poisson brackets is proved in the example of Witten's supersymmetric mechanics.

Of the three known versions of Poisson brackets, two of which are even and one odd with respect to the Grassmann gauge of canonical variables,[1] the odd Poisson brackets, whose canonical variables have the opposite Grassmann gauge, appear more fundamental because this gauge is nontrivial.[2] It is definitely worthwhile to study the various physical applications of odd Poisson brackets.

We wish to call attention to the circumstance that a Hamiltonian system with

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equal numbers of even and odd canonical variables allows the simultaneous introduction of even and odd Poisson brackets. When bracket operations of different gauges are used, the equations for the canonical variables do not change, but the integrals of motion with the opposite Grassmann gauge become duals, converting into each other upon the transformation to the Poisson brackets with the opposite gauge.

We require that the same equations of a dynamic system containing even (x^a and odd x^α) canonical variables be reproduced by even Poisson brackets $\{,\}_0$ with an even Hamiltonian H and by odd Poisson brackets $\{,\}_1$ with an odd Hamiltonian $\overline{H}$. In other words, we require

$$\dot{X}^A = \{X^A, H\}_0 = \{X^A, \overline{H}\}_1 \tag{1}$$

where $X^A = (x^a, x^\alpha)$. Relation (1) is equivalent to the equations

$$\overline{\omega}_{AB}(X)\omega^{BC}\partial_C H = \partial_A \overline{H}, \tag{2}$$

which determine $\overline{H}$ and the coefficients of the closed odd external form

$$\overline{\omega}_{AB}(X) = \partial_A \varphi_B - (-1)^{AB}\partial_B\varphi_A , \tag{3}$$

which corresponds to odd brackets for the given H and to the even canonical form ω^{AB}, where φ_A are coefficients of an odd Liouville form.

To illustrate the point, we seek the solution of Eqs. (2) for the case of Witten's supersymmetric mechanics[3] with the Hamiltonian

$$H = H_0 + i\eta^1\eta^2 W'(q), \tag{4}$$

where $x^a = (q, p)$ and $x^\alpha = (\eta^1, \eta^2)$, and $H_0 = [p^2 + W^2(q)]/2$. For Hamiltonian (4), the fermion charge $F = i\eta^2\eta^2$ and the supercharges $Q_1 = p\eta^1 - W\eta^2$, $Q_2 = p\eta^2 + W\eta^1$, which form a superalgebra with even Poisson brackets

$$\{Q_\alpha, Q_\beta\}_0 = -2i\delta_{\alpha\beta}H, \quad \{F, Q_\alpha\}_0 = \epsilon_{\alpha\beta}Q_\beta . \tag{5}$$

are also conserved quantities. By virtue of equations of motion (1), the quantities H, F, Q_1, and Q_2 and also arbitrary functions of them are also integrals of motion with respect to the odd brackets $\{,\}_1$ with the Hamiltonian $\overline{H}$. Equations (2) with Hamiltonian (4) determine $\overline{H}$ and $\overline{\omega}_{AB}$ within six arbitrary functions that depend on H_0. Making use of this arbitrariness, we can require

$$\overline{H} = Q_1 \tag{6}$$

and that the three other independent quantities, which are conserved with respect to $\overline{H}$ in odd brackets, i.e., the quantities $\overline{F}$, $\overline{Q}_1$, and $\overline{Q}_2$, must be linear in the integrals H, F, Q_1 and Q_2 and must form with odd brackets the superalgebra

$$\{\overline{Q}_\alpha, \overline{Q}_\beta\}_1 = -2i\delta_{\alpha\beta}\overline{H}, \qquad \{\overline{F}, \overline{Q}_\alpha\}_1 = \epsilon_{\alpha\beta}\overline{Q}_\beta ,$$

which is the same as (5). The integrals of motion $\overline{F}$, $\overline{Q}_1$, and $\overline{Q}_2$, are then related in the following way to the conserved quantities in Witten's mechanics with Hamiltonian

(4):

$$\bar{F} = \frac{1}{2i}Q_2\ , \quad \bar{Q}_1 = H\ , \quad \bar{Q}_2 = i(2F-H). \tag{7}$$

Upon the transformation to odd brackets, the supercharges Q_1 and Q_2 acquire the meaning of an odd Hamiltonian $\bar{H}$ and an odd fermion charge $\bar{F}$, while the role of the supercharges $\bar{Q}_1$ and $\bar{Q}_2$ is played by linear combinations of Witten's Hamiltonian (4) and the fermion charge F. By virtue of the symmetry between the charges Q_1 and Q_2 in Witten's mechanics, we could have used Q_2 as the odd Hamiltonian $\bar{H}$.

Under additional conditions (6) and (7), Eqs. (2) have the following solution for the coefficients φ_A of the odd Louiville form:

$$\varphi_q = (1-\alpha_2 W)\eta^1 + \alpha_1 W\eta^2\ ,$$

$$\varphi_p = \frac{1}{W'}[(1+\alpha_1 p)\eta^2 - \alpha_2 p\eta^1],$$

$$\varphi_{\eta^1} = i\eta^1\eta^2\alpha_2\ , \qquad \varphi_{\eta^2} = -i\eta^1\eta^2\alpha_1$$

where

$$\alpha_1 = W'\left[WJ(H_0, q) + \frac{p}{H_0}\right] - \frac{1}{p}\ ,$$

$$\alpha_2 = W'\left[pJ(H_0, q) - \frac{W}{H_0}\right], \qquad J(H_0, q) = \int_{q_0}^{q} [2H_0 - W^2(q')]^{-3/2}dq'\ .$$

We have thus proved that there is an odd Louiville form, determined internally, for Witten's Hamiltonian systems, and we have proved that duality relations (6) and (7) hold between the even and odd integrals of motion; specifically, they hold between the Hamiltonian and the supercharge. The duality relations between the Hamiltonian and the supercharge are of particular interest for relativistic systems, which will be analyzed in a separate paper.

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[3]E. Witten, Nucl. Phys. **B188**, 513 (1981).

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