Helical density waves in flat galaxies–moving solitons

A. B. Mikhaĭlovskiĭ, V. I. Petviashvili, and A. M. Fridman

Institute of Terrestrial Magnetism, Ionosphere, and Radiowave Propagation, USSR Academy of Sciences, Siberian Division

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It is shown that solitons of two types, supersonic and subsonic, can propagate in a rotating gravitating disk. The supersonic solitons are capable of propagating only in the case of a weak Jeans instability of the disk. The subsonic solitons can propagate only in a stable (as defined by Jeans) disk.

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The question of the possible existence of stationary solutions in the form of moving solitons in the plane of a gravitating disk is of exceptional interest both in itself and because of its direct bearing on the problem of the spiral structure of galaxies.

We consider the case of a gravitational instability of an infinitely thin rotating disk, characterized by the following three attributes: a) the maximum instability growth rate γ_{max} is small in comparison with the rotation frequency $\Omega_0(r)$, $\gamma_{max} \ll \Omega_0(r)$ (Fig. 1); b) the size Δk of the wave-number instability region in k-space is small in comparison with a wave number k_0 such that $\gamma_{k0} = \gamma_{max}$, i.e., $\Delta k/k_0 \ll 1$ (Fig. 1); c) the size Δr of the instability region in coordinate space is large in comparison with the wavelength $\lambda_0 = 2\pi/k_0$, $k_0 \Delta r \gg 1$.

These three conditions make it possible to simplify substantially the formal aspect of the solution of the problem, by using standard perturbation-theory methods.

The dispersion equation for small oscillations of a gravitating disk was derived by Toomre[1]; we write it down here with allowance for the azimuthal perturbations

$$\omega^{*2} = \omega_k^2, \tag{1}$$

where

$$\omega^{*2} = (\omega - m\Omega_0)^2, \quad \omega_k^2 = 2\Omega_0 \kappa_0 + k^2 c_s^2 - 2\pi G \sigma^0 |k|. \tag{2}$$

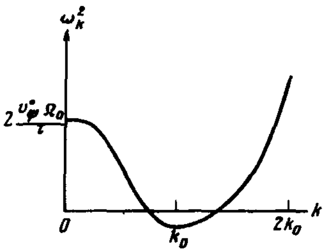

FIG. 1. Case of Jeans instability, $\gamma_{max} = \gamma_{k0}$.

Here G is the gravitational constant, ω is the oscillation frequency

$$\kappa_0 = \frac{1}{r}\frac{\partial}{\partial r}(r v_\phi^0),$$

v_ϕ^0 is the azimuthal velocity of the disk rotation; $k^2 = k_r^2 + k_\phi^2$, but it is assumed here that $k_\phi^2 \ll k_r^2$, k is the modulus of the wave vector, $k_\phi = m/r$; $c_s^2 = \kappa p_0 \sigma_0$, p_0 is the pressure in the plane of the disk, σ_0 is the surface density of the disk, $\kappa = 3 - 2/\gamma$, where γ is the adiabatic exponent: $P/P_0 \sim (\rho/\rho_0)^\gamma$, where P is the usual pressure, ρ is the volume density, $\rho = \sigma\delta(z)$, $\delta(z)$ is the Dirac delta function. The subscript "zero" labels stationary quantities. The condition a) is satisfied if the maximum instability growth rate γ_{max} satisfies the inequality

$$\gamma_{\max} = \sqrt{\pi^2 G^2 \sigma_0^2 / c_s^2 - 2\kappa_0 \Omega_0} \ll \Omega_0. \tag{3}$$

The largest growth of the "fundamental" harmonic k_0 causes the nonlinear effect to initiate the production of overtones with wave numbers $2k_0$, $3k_0$, etc. If all the functions that characterize the disk are expressed as sums of slowly and rapidly varying components, then the rapidly varying part can be expanded in a Fourier series in the fundamental harmonic k_0:

$$X = X^0 + \tilde{X} = X^0 + \sum_n X_n e^{i n k_0 r}, \tag{4}$$

where $X = (v_r, v_\phi, \Psi, \sigma)$, $v_r^0 = 0$, v_r is the radial hydrodynamic velocity, and Ψ is the gravitational potential in the plane of the disk. We start from the system of hydrodynamic equations usually employed in the analysis of perturbations localized in the plane of a gravitating disk ($v_z = 0$). [2]

Substituting the unknown functions in the form (4) in this system of equations and using the indicated three conditions, we express all of the unknown functions in terms of the function of the perturbed azimuthal velocity v_1 (the necessary operation can be found in[4]). For the fundamental harmonic of this function we obtain the following nonlinear differential equation

$$\hat{L}^2 v_1(t,\phi) = \gamma_{k_0}^2 v_1(t,\phi) + \frac{(2-\kappa)k_0^2\Omega_0}{\kappa_0}\left[\frac{8(2-\kappa)\Omega_0\kappa_0}{\omega_{2k_0}^2} - (3-\kappa)\right] \times |v_1(t,\phi)|^2 v_1(t,\phi), \tag{5}$$

where

$$\omega_{2k_0} = m\Omega_0 + \sqrt{2\Omega_0\kappa_0}, \quad \kappa_0 = \frac{1}{r}\frac{\partial}{\partial r}(r v_\phi^0), \quad \kappa_0 > 0, \quad \hat{L} \equiv \frac{\partial}{\partial t} + \Omega_0 \frac{\partial}{\partial \phi}. \tag{6}$$

We shall show that in the region

$$2 - \frac{\omega_{2k_0}^2}{8\Omega_0\kappa_0 - \omega_{2k_0}^2} < \kappa < 2\ . \tag{7}$$

Eq. (5) has a stationary solution of the soliton type.

It is seen from (5) that the growth of the amplitude of the perturbed velocity, due to the instability, is halted at the level

$$|v_1|^2 = \frac{(\gamma_{k_0}^2 + m^2\Omega_0^2)\kappa_0}{(2-\kappa)k_0^2\Omega_0}\left[(3-\kappa) - \frac{8(2-\kappa)\Omega_0\kappa_0}{\omega_{2k_0}^2}\right]^{-1},$$

and the amplitude of the perturbed velocity increases to the value

$$\frac{|\sigma_1|^2}{\sigma_o^2} = \frac{k_r^2}{k_o^2}\frac{(\gamma_{k_o}^2 + m^2\Omega_o^2)}{(2-\kappa)\Omega_o\kappa_o}\left[(3-\kappa) - \frac{8(2-\kappa)\Omega_o\kappa_o}{\omega_{2k_o}^2}\right]^{-1}$$

Assume that a narrow wave packet $\Delta k/k_0 \ll 1$ is excited in the vicinity of the point $\mathbf{k}_0$. It is easy to take into account the scatter of the wave numbers if the function $\gamma^2(\mathbf{k})$ is represented in the form of a series in the vicinity of the point $\mathbf{k}_0$ where this function has a maximum

$$\gamma_{\mathbf{k}}^2 = \gamma_{\mathbf{k}_o}^2 + \frac{1}{2}\frac{\partial^2\gamma_{\mathbf{k}}^2}{\partial k^2}\bigg|_{\mathbf{k}=\mathbf{k}_o}(\mathbf{k}-\mathbf{k}_o)^2 + \dots . \tag{8}$$

As follows from (1),

$$\frac{1}{2}\frac{\partial^2\gamma_k^2}{\partial k^2} \equiv \frac{1}{2}\frac{\partial^2 \mathrm{Im}\,\omega'^2}{\partial k^2} = -c_s^2,$$

$\mathbf{k}-\mathbf{k}_0 = k_r - k_{0r} \equiv k_{1r}$, since $k_r \gg k_\phi = m/r$. We therefore have in place of (8)

$$\gamma_k^2 = \gamma_{k_o}^2 - k_{1r}^2 c_s^2. \tag{9}$$

Replacing in (5) $\gamma_{k_0}^2$ by γ_k^2 and using the expansion (9), we change over in (5) to the coordinate representation. To this end, we multiply term by term (5) by $\exp(ik_1r)$ and integrate with respect to dk_1, assuming that

$$\int v_1(k,t)\, e^{ik_1r}\, dk_1 = v_1(r,t).$$

As a result we obtain the equation

$$v_1(\mathbf{r},t) = (\gamma_{k_o}^2 + c_s^2\Delta_r)\, v_1(\mathbf{r},t) + \frac{(2-\kappa)\,k_o^2\Omega_o}{\kappa_o}\left[\frac{8(2-\kappa)\Omega_o\kappa_o}{\omega_{2k_o}^2} - (3-\kappa)\right] \times |v_1(r,t)|^2 v_1(\mathbf{r}\ t), \tag{10}$$

where

$$\Delta_r \equiv \partial^2/\partial r^2.$$

Choosing the dependence of $v_1(\mathbf{r},t)$ on the coordinates and on the time in the form $v_1(\mathbf{r},t) = V(\xi) \equiv V(k_r r + m\phi - \omega t)$, we change over in (10) to the variable ξ:

$$a^2\frac{d^2V}{d\xi^2} = \gamma_o^2 V - \beta^2 V^3,$$

where

$$a^2 = (\omega - m\Omega_o)^2 - k_r^2 c_s^2, \quad \beta^2 = \frac{(2-\kappa)\,k_o^2\Omega_o}{\kappa_o}\left[(3-\kappa) - \frac{8(2-\kappa)\Omega_o\kappa_o}{\omega_{2k_o}^2}\right] \tag{12}$$

The solution of (11) is given by

$$V(\xi) = \frac{a}{b}\frac{1}{\mathrm{ch}\, a\xi}, \tag{13}$$

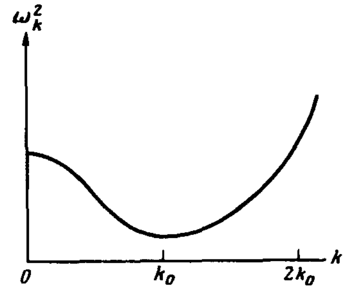

FIG. 2. Case of a disk that is stable (as defined by Jeans), $\gamma = 0$.

$$a = \gamma_0^2/a^2, \quad 2b^2 = \beta^2/a^2. \tag{14}$$

Thus, as seen from formulas (11)—(14), two types of solitons can exist in a rotating gravitating gas disk.

1) A supersonic soliton, $\alpha^2 > 0$, which can set in and propagate in a weakly unstable disk, $\gamma_{k_0}^2 > 0$ (Fig. 1). The equation of the state should satisfy in this case the condition $\beta^2 > 0$. Since $\kappa > 0$ [see formula (6)], the condition $\beta^2 > 0$ corresponds to the following: to an adiabatic exponent γ in the region $\frac{3}{2} < \gamma < 2$ in the case of an arbitrary function $\kappa_0^2 > 0$ and a mode $m = 0$; in the case of a rigid-body-rotation disk: a) to $\frac{5}{4} < \gamma < 2$ for the mode $m = 1$ and b) to all reasonable values of γ for the mode $m \geq 2$.

2) A subsonic soliton, $\alpha^2 < 0$, which can propagate in a disk that is stable (as defined by Jeans), $\gamma_k^2 < 0$ (Fig. 2). The equation of state of the disk must satisfy in this case the condition $\beta^2 < 0$, corresponding to an adiabatic exponent $\gamma < \frac{3}{2}$, in the case of an arbitrary function $\kappa_0^2 > 0$ and the mode $m = 0$, and to $\gamma < \frac{5}{4}$ in the case of a disk rotating as a rigid body and the mode $m = 1$.

[1]A. Toomre, Astrophys. J. 139, 1217 (1964).
[2]V. L. Polyachenko and A. M. Fridman, Ravnovesie i ustoĭchivost' gravitiruyushchikh sistem (Equilibrium and Stability of Gravitating Systems), Nauka, 1976.
[3]C. Hunter, Ann. Rev. Fluid Mech. 4, 219 (1972).