Width of the spectrum of nearly-harmonic self-oscillating systems

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The question of the distribution (on account of natural fluctuations) of the average energy of a physical, chemical, or biological self-oscillating system is of fundamental significance. In particular, the line width determines the sensitivity limit of many instruments.

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The calculation of the spectrum of self-oscillating systems has been the subject of an extensive literature (see, e.g., [1–3]). Owing to the nonlinearity of the initial Langevin equations, this problem is solved approximately. In the calculation one can introduce two time-dependent parameters τ_A and $\tau_\phi \equiv 1/D_\phi$. The first ($\tau_A$) determines the relaxation time of the amplitude A, while the second (τ_ϕ) determines the relaxation time of the phase. D_ϕ is the diffusion coefficient of the phase. In the advanced-generation regime we have $\tau_\phi \gg \tau_A$, so that we can introduce a small parameter $\varepsilon = \tau_A/\tau_\phi$.

In the zeroth approximation in ε (without allowance for the amplitude fluctuations), the spectrum is a Lorentz line with half-width $\Delta\omega = D_\phi$, and consequently the line width is completely determined by the phase diffusion D_ϕ. In first-order approximation in ε, the spectrum consists of a sum of two Lorentz lines, one narrow and one broad (pedestal). The half-width of the narrow line is now $D_\phi(1+\frac{3}{4}\varepsilon) \gtrsim D_\phi$, and that of the broad one is $4D_\phi/\varepsilon \gg D_\phi$.

When account is taken of higher approximations in $\varepsilon = \tau_A/\tau_\phi$, accumulation of the contributions of the amplitude fluctuations becomes possible, and the problem arises of determining the resultant width of the spectral line.

It is shown in the present paper that the effect of accumulation of the contributions of the amplitude fluctuations makes the width of the resultant Lorentz line in the advanced-generation regime equal to $2D_\phi$ and not D_ϕ, and consequently the phase diffusion does not determine the line width completely. This changes our notions concerning the factors that govern the line width of the spectrum of self-oscillating systems.

In place of the Langevin equations for the amplitude and phase,[1–3] we start from the equations for the equal-time correlation

$$\langle E(\tau)\rangle = \frac{1}{2} \langle A(t)A(t-\tau)\cos[\phi(t)-\phi(t-\tau)]\rangle \tag{1}$$

and the average energy $\langle E\rangle$. At $\tau=0$ the function (1) is equal to $\langle E\rangle$, so that the function E_ω is the distribution of the average energy over the spectrum and $\langle E\rangle = (1/2\pi)\int_{-\infty}^{\infty} E_\omega d_\omega$.

From the Fokker-Planck equation for the two-time distribution function at cubic nonlinearity we obtain the equations for the function (1)

$$\frac{d}{d\tau}<E(\tau)> + \frac{1}{2}<(\gamma+\eta E)E(\tau)> = 0. \qquad (2)$$

We need also an equation for the mean values of the energy E and the amplitude A. They also follow from the Fokker-Planck equation and assume in the stationary state the form

$$<(\gamma+\eta E)E> = D, \qquad <(\gamma+\eta A^2/2)A> = D<1/A>. \qquad (3)$$

D is the intensity of the Raman source in the Langevin equations. In the case of action of thermal noise $D=\gamma kT$.

Equations (2) and (3) are not closed, inasmuch as they contain, owing to the nonlinearity of the initial Langevin equations, not only the first but also higher moments. For a linear harmonic oscillator, when $\eta=0$ and $\gamma>0$, it follows from (2) and (3) that the line half-width is $\Delta\omega=\gamma/2$ and the average energy is $\langle E\rangle=D/\gamma=kT$.

To calculate the spectrum of a self-oscillating system it is necessary, in the considered approach, to solve approximately a system of nonlinear equations for the moments. We obtain here the sought line width on the basis of the following reasoning.

We represent the function (1) in the form

$$<E(\tau)> = \sum_i <E>_i e^{-\lambda_i\tau} = \sum_i <E>_i e^{-\lambda\tau} = <E>e^{-\lambda\tau}. \qquad (4)$$

This is possible if the spectrum is represented as a sum of Lorentz lines with half-widths $(\Delta\omega)_i=\lambda_i$. From (2) and (4) we obtain an equation for λ:

$$\lambda = \frac{1}{2<E>}<(\gamma+\eta E)E(\tau)> e^{\lambda\tau}. \qquad (5)$$

The right-hand side of this equation is independent of τ if

$$<(\gamma+\eta E)E(\tau)> = Be^{-\lambda\tau}. \qquad (6)$$

The constant B is to be determined. We obtain it for two limiting cases. The first corresponds to the zeroth approximation in ε, i.e., to total neglect of the amplitude fluctuations. In this case we can make the substitution $A(t-\tau) \to \langle A\rangle$ under the $\langle\rangle$ sign in (5), i.e., we can neglect the correlation of the factor $A(t-\tau)$ with the remaining factors. It is necessary here to retain in (4) only one term with the smallest λ_i, and the initial term in (6) should be chosen to be the moment after the attenuation of the fast fluctuations. From (6) and (3) we get in this approximation

$$B = <(\gamma+\eta A^2/2)A>\frac{<A>}{2} = \frac{1}{2}D<\frac{1}{A}><A> = \frac{1}{2}D. \qquad (7)$$

We have used here the second equation of (3) and the condition for the developed generation ($\delta A \ll \langle A\rangle$). From (5)—(7) we get

$$(\lambda)_0 \equiv (\Delta\omega)_0 = \frac{D}{4<E>_0} = \frac{D}{2<A>_0^2} \equiv D_\phi <E>_0 = \frac{|\gamma|}{\eta}. \qquad (8)$$

In the second limiting case we take into account in (4) all the terms of the series, which corresponds to taking account of the total correlation of the factor $A(t-\tau)$ with the remaining factors in (5) and (6). The constant B in (6) is then equal to the left-hand side of (6) at $\tau=0$, i.e.,

$$B = \langle(\gamma+\eta E)E\rangle = D, \quad \lambda \equiv \Delta\omega = \frac{D}{2\langle E\rangle_0} = 2(\Delta\omega)_0. \tag{9}$$

We have here the first equation of (3).

We note that we can write for $(\Delta\omega)_0$ and $\Delta\omega$ formulas that are valid for any excess above the threshold

$$(\Delta\omega)_0 = \frac{D}{2\langle A\rangle}\left\langle \frac{1}{A}\right\rangle, \qquad \Delta\omega = \frac{D}{2\langle E\rangle}. \tag{10}$$

From this we get in the advanced generation regime $\Delta\omega = 2(\Delta\omega)_0$, and in the equilibrium state ($\eta=0$, $\gamma>0$) $(\Delta\omega)_0 = \Delta\omega$, since in this case $\langle A\rangle\ \langle A^{-1}\rangle = \langle E\rangle$. Thus, the two curves merge as equilibrium is approached.

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