Summation of asymptotic series in quantum field theory

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A method is considered for summing perturbation-theory series with factorially increasing coefficients a_k. A connection is established between the asymptotic value of a_k as $k \to \infty$ and the character of the singularity of the sum of the series. The limits of the region of applicability of the improved perturbation theory are obtained. The results were verified for a number of physical problems for which exact solutions are known.

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The effective methods of calculating high orders of perturbation theory have been discovered recently[1,2] in quantum mechanics (the case particularly investigated is that of the anharmonic oscillator).[2] Lipatov[3,4] has developed a quasiclassical method of calculating the functional integral in quantum field theory and found the asymptotic form of the coefficients of a number of perturbation theory (PT) series for the Gell-Mann—Low functions (GLF) in scalar theory with interaction $H_{\text{int}} = g\int \phi^n d^D x/n!$, $D = 2n/(n-2)$. This trend is developing intensively (see, e.g.,[5–9]). The present article deals with the question of how much information concerning the behavior of the exact solutions can be extracted from the asymptotic forms of the coefficients a_k.

Let $f(z)$ be a function represented by an asymptotic (as $z \to 0$) series:

$$f(z) = \sum_{k=k_0}^{\infty} a_k(-z)^k. \tag{1}$$

In most of the considered theories[2-6] the coefficients are given by

$$a_k = (ka)!\ a^k k^{\beta}(c_0 + \frac{c_1}{k} + \frac{c_2}{k^2} + \dots), \tag{2}$$

where

$a,\ a > 0,\ z! \equiv \Gamma(z+1).$

The more convenient representation of the asymptotic form of a_k as $k \to \infty$ is:

$$a_k = \frac{(ka)!}{k!} a^k \sum_{m=0}^{\infty} C_m(k + \beta - m)!. \tag{3}$$

The coefficients of these two series are connected by the linear transformation

$$c_i = \sum_{j=0}^{i} S_{ij} C_j.$$

For the elements of the matrices S and S^{-1} we have obtained explicit formulas for arbitrary i and j, making it possible to change from representation (3) to (2) or vice versa. Substituting (3) in (1) and using Borel's method, we obtain the sum of the series (1):

$$f(z) = (az)^{-(\beta+1)} e^{1/az} \sum_{m=0}^{\infty} (\beta - m)!\ C_m (az)^m J(az; a, m\beta), \tag{4}$$

where

$$J(x; a, \beta) \equiv x^{-\beta} \int_1^{\infty} e^{-t/x} [(t-1)^a + 1]^{\beta - 1} dt.$$

In the frequently encountered case $\alpha = 1$, the integral reduces to an incomplete Gamma function $J(x;\ 1, \beta) = \Gamma(\beta, x^{-1})$. The same result is obtained by summing the series (1) with the coefficients (3) by the Sommerfeld-Watson method, as proposed in[2]. The possibility of obtaining the answer in closed form convenient for numerical calculations is the advantage of (3) over the parametrization (2).

The point $z = 0$ is a branch point for $f(z)$. Following,[10] we can show that the asymptotic form of a_k determines the character of the singularity of the sum of the series (1), namely, the behavior of the jump of the function $f(z)$ on the cut $z < 0$ as $z \to -0$:

$$\Delta f(z) = \pi C_0 a^{-(\beta+1)} (-az)^{-(\beta+1)/a} \exp\{-(-az)^{-1/a}\}. \tag{5}$$

In field-theory problems, only the first term of the expansion (2) can so far be calculated; we denote it by a_k. Several exact coefficients a_k of the PT series can be obtained by calculating Feynman diagrams. This enables us to construct an improved PT:

$$f_N(z) = \tilde{f}(z) + \sum_{k=k_0}^{k_0+N-1} (a_k - \tilde{a}_k)(-z)^k, \tag{6}$$

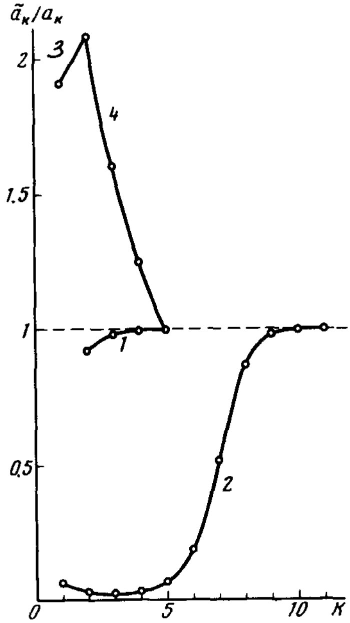

FIG. 1.

where $\tilde{f}(z)$ is the Borel sum of the series $\Sigma\tilde{a}_k(-z)^k$. For the function $f_N(z)$, the first N coefficients coincide with the exact ones, and account is also taken of the remote "tail" of the PT series. It is natural to expect the improved PT to approach the exact solution with increasing N. An investigation has shown that this is indeed the case, but only in the region $0 < z < z_N$:

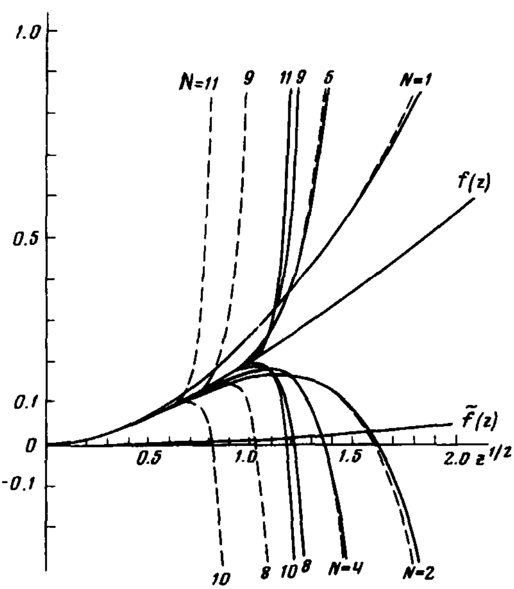

FIG. 2.

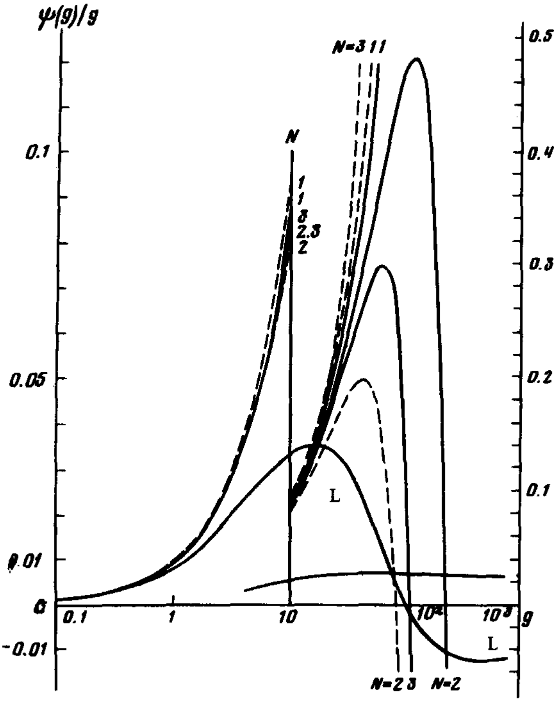

FIG. 3.

$$z_N \approx \frac{1}{a}\left(\frac{e}{Na}\right)^a; \quad N >> 1, \; e = 2.718\ldots . \tag{7}$$

At $z > z_N$, the functions $f_N(z)$ depart steeply from the exact solution. These results were verified for a number of physical problems for which the PT series take the form (2) and exact solutions can be obtained: 1) the Heisenberg-Euler Lagrangian in the case of constant and homogeneous fields E and H; 2) the electron level energies at $z > 137$ (see[11]); 3) the relativistic Thomas-Fermi equation for vacuum screening[12]; 4) the Stark effect in a strong electric field. For lack of space we present only part of the results. Figure 1 shows how the asymptotic form evolves (the curves are labeled with the number of the problem). The relation between the improved PT and the exact solution is illustrate(using problem 2 as an example. The energy ε of the $1s_{1/2}$ level near the boundary of lower continuum is determined by the equation $f(z) = (\zeta^2 - 1)^{-1/2} - (\zeta_{cr}^2 - 1)^{-1/2}$, where $\zeta = z$, $z = \varepsilon^{-2} - 1$, and $f(z)$ is given in[11]. Figure 2 shows this function, as well as the PT polynomials (dashed curves; the numbers indicate the degree of the polynomials) and $f_N(z)$ (solid curves). The transition from the PT to the improved PT extends its applicability region, a fact particularly noticable at large N. Where the neighboring curves of the improved PT are close to one another, they approximate well the exact curves. This takes place also for problems 1, 3, and 4.

Let us apply this method to the scalar $g\,\phi^4/4!$ theory, for which the GLF coefficients a_2, a_3, and a_4 and the asymptotic a_k are known[3,4]; this enables us to calculate $\tilde{\psi}(g)$ and $\psi_N(g)$ at $N = 1, 2, 3$. These curves (see Fig. 3) are close to one another at the $g \lesssim 40$,[1)] so that we can conclude that the exact function $\psi(g)$

is also close to them. On the other hand, at $g=10-40$ the functions $\psi_N(g)$ differ by a factor 3—10 from the curve L obtained in[3]. This attests to the poor accuracy of the expansion in $1/n$, as a result of which the conclusion[3] that the GLF have zeroes seems doubtful (see also[8]). All that has been reliably established is the jump of the GLF on the cut in the vicinity of zero. For a scalar theory of the type $g\ \phi^n/n!$ we have

$$\Delta\psi(g) = A_1|g|^{-(n^2+n-2)/(n-2)^2}\exp(-A_2|g|^{-2/(n-2)}), \tag{8}$$

where $g\rightarrow -0$ and A_1 and A_2 are constants that depend on n.

The situation with the zeros of the GLF may become clear if the next higher coefficients of PT series are calculated and a method is obtained for reconstructing the function $\psi(g)$ in a larger inteval of g than the improved PT (6). A possible candidate is the Padé-approximant method.

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1)At $g>10$ the scale for curves 1—3 is changed by a factor of four and is shown on the right.

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