Exact solution of the Ising model on a random two-dimensional lattice

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The Ising model on a set of plane lattices with a coordination number of 4, over which a summation is a new dynamic degree of freedom, reduces to an exactly solvable two-matrix model in the limit $N \to \infty$. At a temperature $T_c = 1/\ln 2$, a third-order phase transition occurs in the model. This transition corresponds to a spontaneous ordering of the spins. There is a formal correspondence between this model and a zero-dimensional fermion string.

The two-dimensional Ising model and its various generalizations have generated many useful ideas in quantum field theory and the physics of phase transitions. As a rule, however, the corresponding studies have been restricted to the choice of one of the various regular two-dimensional lattices.

In the present letter we derive an exact expression for the partition function $Z_n(\beta)$ of the Ising model on a set of arbitrary plane lattices with the topology of a sphere and with n vertices in the thermodynamic limit, $n \to \infty$:

$$Z_n(\beta) = \sum_{\{G^{(n)}\}} \sum_{\{\sigma\}} \exp[\beta G_{ij}^{(n)} \sigma_i \sigma_j], \tag{1}$$

where $\sigma_i = \pm 1, i = 1,...,n$ are the Ising spins; $G_{ij}^{(n)}$ is the connection matrix of the plane graph (the matrix of nearest neighbors, of which there are always four), given by

$$G_{ij}^{(n)} = \begin{cases} 1, & \text{if } \sigma_i \text{ and } \sigma_j - \text{ or the adjacent spins on the graphs,} \\ 0, & \text{in the other cases,} \end{cases} \tag{2}$$

$\beta = 1/T$ is the inverse Ising temperature (we will consider only the ferromagnetic case here); $\Sigma_{\{\sigma\}}...$ is a summation over all spin configurations; and $\Sigma_{\{G^{(n)}\}}...$ is a summation over all plane graphs with the topology of a sphere and with n vertices. One spin-lattice configuration is shown in Fig. 1.

We can show that the generating function for partition function (1) is of the form

$$Z(\beta, g) = \sum_{n=1}^{\infty} \left[\frac{-4gc}{(1-c^2)^2}\right]^n Z_n(\beta), \tag{3}$$

where

$$c = e^{-2\beta} \tag{4}$$

is (within a g-independent contribution) the free energy of the model of two Hermi-

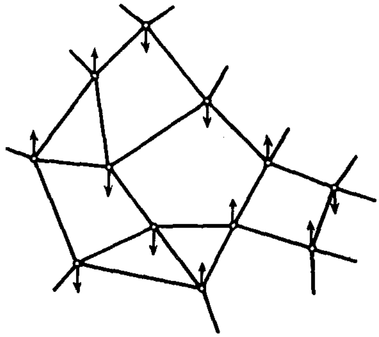

FIG. 1.

tian matrices u_{ij} and v_{ij}, $i, j = 1,\ldots, N$:

$$F(c, g) = \ln \int du \int dv \exp \mathrm{tr}\,[-u^2 - v^2 + 2cuv - gu^4 - gv^4] \qquad (5)$$

in the plane limit,[1,2] $N \to \infty$.

For a proof it is sufficient to note that a series expansion of $F(c,g)$ in the coupling constant g is described by a diagram technique of the theory of the type φ^4 with two types of vertices, u-vertices and v-vertices, and, correspondingly, with two types of propagators,

$$\langle \frac{\mathrm{tr}}{N} u^2 \rangle = \langle \frac{\mathrm{tr}}{N} v^2 \rangle = \frac{1}{1 - c^2}, \qquad (6)$$

$$\langle \frac{\mathrm{tr}}{N} uv \rangle = \frac{c}{1 - c^2}. \qquad (7)$$

We associated with a given diagram an equivalent lattice, at whose vertices we place Ising spins (Fig. 1). The u- and v-vertices are associated with spins directed upward and downward, respectively. The energy of the coupling of two adjacent spins is given by

$$\exp(\beta E_{\uparrow\uparrow}) = \exp(\beta E_{\downarrow\downarrow}) = \frac{1}{1 - c^2}, \qquad (8)$$

$$\exp(\beta E_{\uparrow\downarrow}) = \frac{c}{1 - c^2}. \qquad (9)$$

In the limit $N \to \infty$, we of course find only plane diagrams with the topology of a sphere. The logarithm in (5) must be used in order to select exclusively coupled lattices. The equivalence of (3) and (5) is thus verified directly in each order in g.

The model proposed here allows a duality transformation analogous to a transformation on a regular square lattice. We find an Ising model on a set of dual plane graphs $\{G^*\}$ constructed of tetragons. The dual temperature is related to the original temperature by

$$\tilde{\beta} = -\tfrac{1}{2} \ln \tanh \beta. \qquad (10)$$

The condition $\tilde{\beta} = \beta$, however, does not determine the critical point, since the model is not self-dual.

It is possible to establish a formal correspondence between the local limit of our model ($n \to \infty$, with lattice constant $a \to 0$) and a zero-dimensional model of a string with interior fermions. This correspondence, which of course must be supported by dynamic arguments of some sort, follows from the two assertions that (1) the Ising model on an arbitrary plane lattice is equivalent to a model of free interior fermions on this lattice[3] and (2) a plane lattice formally describes a curved two-dimensional space and can serve as a regularization of the world surface of a string.[4]

The fact that we are dealing with something other than a triangular lattice is not of major importance, since all these models appear to belong to a common universality class.

We can now write an exact expression for the partition function $Z_n(\beta)$ in the limit $n \to \infty$. From the results of Ref. 2, we can easily extract the following parametric representation of $F(c,g)$:

$$F(c,z) = \frac{1}{2}\ln\frac{z}{g(z)} + \frac{z^2}{2g^2(z)}\left[\frac{1}{2}\frac{z-1}{(3z-1)^3} + c^2\frac{z+1}{3z-1} + \frac{c^4}{2}(3z^4 - 3z^2 + 1)\right]$$
$$- \frac{z}{g(z)}\left[\frac{1}{3z-1} + c^2(1-z^2)\right] + \frac{1}{2}\ln(1-c^2) + \frac{3}{4}, \tag{11}$$

$$g(z) = \frac{z}{(3z-1)^2} - c^2 z + 3c^2 z^3 \cdot \tag{12}$$

Expression (11) was derived through a direct evaluation of the integrals in the final result of Ref. 2 for $F(c,g)$. It is convenient to introduce the new variable $z = 2gf/c$, where $f(x)$ is a function used in Ref. 2.

The singularities of the function $F(c,g)$, which determine the asymptotic form of the expansion coefficients in Ref. 3, can be found from the equation

$$g_z'(z) = 0. \tag{13}$$

The roots of this equation are given by

$$z_0 = -\frac{1}{3}, \tag{14}$$

$$z_{1,2} = \frac{1}{3}\left(1 \mp \frac{1}{\sqrt{c}}\right), \tag{15}$$

$$z_{3,4} = \frac{1}{3}\left(1 \mp \frac{i}{\sqrt{c}}\right). \tag{16}$$

At temperatures in the physical interval $0 \leqslant c \leqslant 1$, only the roots z_0 and z_1 can determine the asymptotic behavior of the expansion of $F(c,g)$. The corresponding critical coupling constants,

$$g_0 = -\frac{1}{12} + \frac{2}{9}c^2 , \qquad (17)$$

$$g_1 = -\frac{2}{9}\sqrt{c}(\sqrt{c}-1)^2(\sqrt{c}+2), \qquad (18)$$

are equal to each other in the case $c = 1/4$. It is easy to see that by varying the parameter z from 0 to $-\infty$ with $0 \leqslant c < 1/4$ we encounter the singularity g_0 first, while g_1 remains on the second, nonphysical sheet of the function $F(c,g)$. The asymptotic form of the expansion coefficient in (5) in this interval is determined by the relation

$$Z_n(c) \simeq n^{-b}\left[-\frac{4cg_0(c)}{(1-c^2)^2}\right]^{-n}, \qquad 0 \leqslant c \leqslant 1/4. \qquad (19)$$

For $1/4 < c \leqslant 1$, the singularities g_0 and g_1 exchange roles, and we find the different asymptotic behavior

$$Z_n(c) \simeq n^{-b}\left[-\frac{4cg_1(c)}{(1-c^2)^2}\right]^{-n}, \qquad 1/4 \leqslant c \leqslant 1. \qquad (20)$$

In the limit $n \to \infty$, we thus find the final result for the partition function of the Ising model on a random lattice:

$$[Z_n(c)]^{-1/n} \simeq \begin{cases} -\dfrac{4cg_0(c)}{(1-c^2)^2}, & \text{for } 0 \leqslant c \leqslant 1/4 \\ -\dfrac{4cg_1(c)}{(1-c^2)^2}, & \text{for } 1/4 \leqslant c \leqslant 1 \end{cases}. \qquad (21)$$

The critical temperature corresponds to $c = 1/4$:

$$\beta^* = \ln 2 . \qquad (22)$$

Now calculating the derivatives $\partial^k Z(c)/\partial c^k$, using (17), (18), and (21), we find that $\partial Z/\partial c$ and $\partial^2 Z/\partial c^2$ are continuous at the transition point, while $\partial^3 Z/\partial c^3$ has a finite discontinuity; this situation corresponds to a third-order phase transition. As the temperature at the transition point is raised, a spontaneous ordering of spins occurs in the system, as can be seen from a comparison of the low-temperature and high-temperature asymptotic expressions:

$$Z_n(c)_{c\to 0} \sim (12/4\cdot c)^n , \qquad (23)$$

$$Z_n(c)_{c\to 1} \sim (24/4)^n . \qquad (24)$$

The low-temperature limit in (23) does not differ from the asymptotic form of the expansion coefficients for a single-matrix model of the φ^4 type[5]; this situation corresponds to Ising spins that have "frozen." The difference (by a factor of $2c$) in the number which is raised to the power in the high-temperature case, (24), corresponds to a summation over completely disordered spins.

It is also easy to see that (11) implies

$$F'_z(z_0) = F''_{zz}(z_0) = F'_z(z_1) = F''_{zz}(z_1), \tag{25}$$

which we can use, along with (13), to find the value of the index b in (19) and (20) for $c \neq c^*$:

$$b = 7/2, \quad c \neq 1/4 \tag{26}$$

This result is the same as that for the index b in the single-matrix model.[5]

With $c = c^*$ we find the relation $g''_{zz}(z) = 0$, in addition to (25) and (13), so that the index b changes:

$$b^* = 10/3, \quad c = 1/4. \tag{27}$$

We see that at the critical point the fermions become long-range entities and affect fluctuations of the "geometry" of the lattice.

The correspondence which we have pointed out here could also be used to study the Ising model on a random graph in an arbitrary magnetic field and model of the same type with symmetry Z_3. We will take up those questions and also a generalization to a triangular lattice in an expanded paper.

[1]C. Itzykson and J.-B. Zuber, J. Math. Phys. **21**, 411 (1980).
[2]M. L. Mehta, Commun. Math. Phys. **79**, 327 (1981).
[3]M. A. Bershadskiĭ, I. D. Vaĭburd, and A. A. Migdal, Pis'ma Zh. Eksp. Teor. Fiz. **43**, 153 (1986) [JETP Lett. **43**, 193 (1986)]; A. A. Migdal, Phys. Rept. **102(4)**, 199 (1983).
[4]V. A. Kazakov, Phys. Lett. **150B**, 282 (1985); F. David, Nucl. Phys. **B257**, 45 (1985); J. Ambjorn, B. Durhuus, and J. Fröhlich, Nucl. Phys. **B257**, 433 (1985); V. A. Kazakov, I. K. Kostov, and A. A. Migdal, Phys. Lett. **157B**, 295 (1985).
[5]E. Brezin, C. Itzykson, G. Parisi, and J.-B. Zuber, Commun. Math. Phys. **59**, 35 (1978).

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