Stochastic nature of spherically symmetric solutions of the time-dependent Yang-Mills equations

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(Submitted 29 April 1986)
Pis'ma Zh. Eksp. Teor. Fiz. 44, No. 3, 109–111 (10 August 1986)

The spherically symmetric time-dependent Yang-Mills equations are shown to be a nonintegrable system. In particular, the phase space near the Wu-Yang static solution is a stochastic region.

The classical Yang-Mills equations, which depend on only the time ("classical Yang-Mills mechanics"), are known to be a nonintegrable system.[1–3] It is time for an analysis of a general $(3+1)$-dimensional Yang-Mills field system from this standpoint.

In this letter we present the results of such a study for spherical symmetry, in which case the non-Abelian vector potential A_μ^a depends on $r = |\mathbf{r}|$ and t. At the same time, we find an answer to the question of the stability of the known spherically symmetric static solutions of the Wu-Yang type.[4]

Our problem reduces to a study of a nonlinear-string equation

$$u_{tt} - u_{xx} = F(x, u, u_x, u_t, \dots), \tag{1}$$

where u_t, u_{tt}, etc., are the derivatives with respect to the corresponding argument. In general, the question of the integrability of such a system cannot be studied analytically.

The best approach here is that proposed in the well-known paper[5] by Fermi, Pasta, and Ulam: to replace the continuous string in (1) by its discrete analog—a chain of coupled anharmonic oscillators—and to observe how the energy of its vibrations is distributed among harmonics. In the general case, in which the system is nonintegrable, there is a pumping of energy among several harmonics in the case of small initial perturbations, and the system undergoes an anomalously slow "thermalization," as observed by Fermi *et al.*[5] For large perturbations, in contrast, the motion occurs in an ergodic layer, with the consequence that the energy becomes distributed equally among the harmonics, as is observed in the Fermi-Pasta-Ulam system.[6] It is this approach we have taken for our study of spherically symmetric Yang-Mills equations. The general form of A_μ^a for these equations, for the SU(2) group in a $(3+1)$-dimensional space, is given by[7]

$$A_j^a = \frac{\varphi_1}{r^3}(\delta_{ja} r^2 - x_j x_a) + \frac{1+\varphi_2}{r^2}\epsilon_{jak} x_k + A_1 \frac{x_j x_a}{r^2}, \tag{2}$$

$$A_0^a = A_0 x_a / r,$$

where the arbitrary functions $\varphi_{1,2}$ and $A_{0,1}$ depend on r and t. We can set $A_0 = 0$ by

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virtue of gauge invariance. Since we will be examining in detail the special case $\varphi_1 = A_1 = 0$, we will be dealing with a nonlinear string (1) of the type ($\varphi \equiv \varphi_2$):

$$(\partial_t^2 - \partial_r^2)\varphi = -\frac{1}{r^2}\varphi(\varphi^2 - 1). \tag{3}$$

The static solution of (3), $\dot{\varphi} = 0$, is the Wu-Yang monopole

$$A_j^a = \frac{1}{r^2}\epsilon_{jak}x_k,$$

where $\varphi = -1$ is the vacuum solution $A_j^a = 0$, and $\varphi = 1$ gives a field which is gauge-equivalent to the vacuum field.

A qualitative analysis of Eq. (3) shows that the spherically symmetric solutions of the Yang-Mills equations are unstable even in the static limit ($\pi_\varphi = \partial\varphi/\partial t = 0$): Small changes in the initial conditions [$\varphi(r)$ and $\partial\varphi/\partial r$] causes dramatic changes in the behavior of the solutions, giving rise to or changing in the positions of singularities. We can distinguish five static solutions of (3) which remain finite at all $r \geqslant 0$. These are the solutions $\varphi = \pm 1$ and $\varphi = 0$, which are already known, and the two separatrices of Eq. (3), $\varphi_{C_{1,2}}(r)$, which become the Wu-Yang solution in the limit $r \to 0$, while in the limit $r \to \infty$ they become the vacuum solutions [$\varphi(r) = \pm 1$]. To the best of our knowledge, the solutions $\varphi_{C_{1,2}}(r)$ have not been recognized previously.

Analysis of the phase paths of system (3) near the static solutions reveals an exponential temporal instability for the solutions $\varphi(r) = 0$ and for the new solutions $\varphi_{C_{1,2}}(r)$. It also reveals a stability of the vacuum solutions $\varphi(r) = \pm 1$ with respect to small perturbations. All the solutions of (3) which are singular (with $\pi_\varphi = 0$) are also exponentially unstable.

We studied finite perturbations of (3) in numerical simulations like those of Ref. 5. The continuous string in (3) was approximated by a set of coupled nonlinear oscillators $(\varphi)(i)$ ($i = 1, 2, \ldots, N$), whose number N was chosen to be 64 or 128.

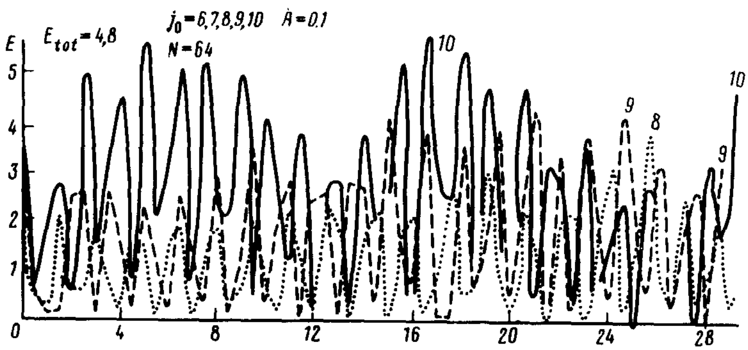

FIG. 1.

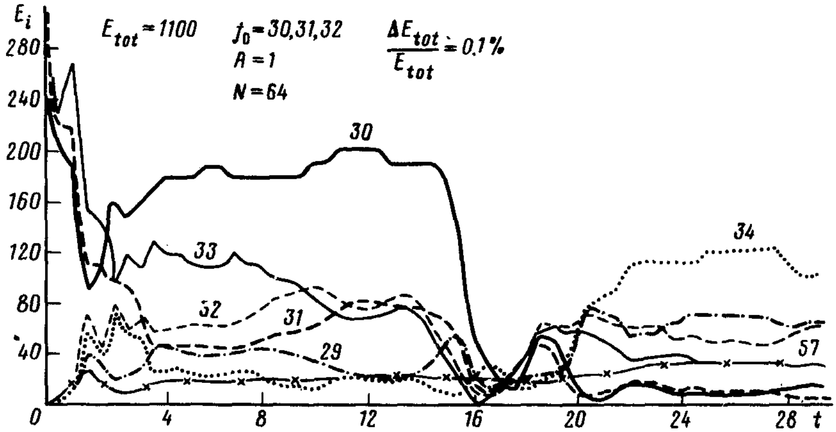

FIG. 2.

We numerically integrated a discrete analog of Eq. (3):

$$\ddot{\varphi}(i, t) = \frac{\varphi(i+1, t) - 2\varphi(i, t) + \varphi(i-1, t)}{(\Delta r)^2} - \frac{\varphi(i, t)[\varphi^2(i, t) - 1]}{(i\,\Delta r)^2} \tag{4}$$

(Δr is the quantization step, which we chose to be 0.1). We studied perturbations near the static solutions $\varphi = 0$, $\varphi = \pm 1$, $\varphi_{C_{1,2}}(r)$ by introducing corresponding initial and boundary conditions expressed as harmonic expansions:

$$\varphi(i, t) = \sqrt{2/N} \sum_{j=1}^{N-1} \psi(j, t) \sin(\pi i j/N).$$

Figure 1 shows some representative curves of the energies of three modes ($j = 8,9,10$) versus the time for a small initial perturbation of five of the modes (with an amplitude $A = 0.1$): $j_0 = 6$–10. Here $N = 64$. The total energy of the "string" in (4) is $E_{\rm tot} = 4.8$ ($\Delta E_{\rm tot}/E_{\rm tot} < 1\%$). The $j_0 = 6,7$ modes behave in an analogous way, while all the other modes remain essentially unexcited. We see that the system executes an approximately quasiperiodic motion.

The picture changes substantially as the energy of the string is increased. Figure 2 ($N = 64, E_{\rm tot} = 1100$, $\Delta E_{\rm tot}/E_{\rm tot} = 0.1\%$, $j_0 = 30,31,32$) clearly shows how the ener-

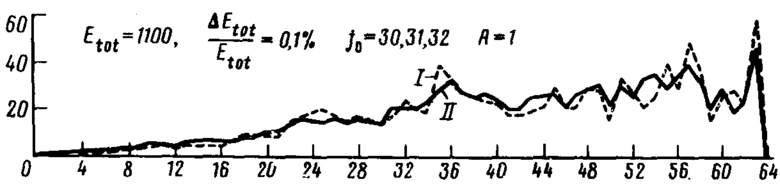

FIG. 3.

gies of the modes initially excited ($j_0 = 30$–32; with an amplitude $A = 1$) and of the other modes become equalized (the energies of only some of the modes, $j = 29, 33, 34$, and 57, are shown in Fig. 2). A further (threefold) increase in the energy in these modes leaves the picture qualitatively the same.

The system undergoes a thermalization (transition to chaos). As we increase N (we took N to be 128), i.e., as we approach the continuous limit, as can be seen from an examination of Eq. (4), the string "thermalizes" with smaller perturbations.

That the system is stochastic is also implied by Fig. 3, which shows the distribution of the energy among modes, averaged over a long time (the averaging interval was 780 at a step of 0.04): $\overline{E}_i$ ($i = 1, 2, \ldots, 64$). We see that, on the average, all of the modes are excited.

In summary, not only classical Yang-Mills mechanics[1,2] but also the classical Yang-Mills field theory describing a system with an infinite number of degrees of freedom is nonintegrable, i.e., exhibits a dynamic chaos.

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Translated by Dave Parsons