Vanishing of the vacuum energy for superstrings

A. Yu. Morozov and A. M. Perelomov
Institute of Theoretical and Experimental Physics

(Submitted 10 July 1986)
Pis'ma Zh. Eksp. Teor. Fiz. **44**, No. 4, 157–160 (25 August 1986)

A hypothesis regarding the structure of the equations for the vacuum diagrams in a first-quantized theory of superstrings is proposed. The analytic measure in the integral over modulus space is proportional to the sum $\Sigma\epsilon_e\theta[e]^4$ ($\epsilon_e = \pm 1$) over spin structures on Riemann surfaces and vanishes by virtue of the Riemann identities for θ constants.

1. Recent work based on the Belavin-Knizhnik theorem regarding the cancellation of the analytic anomaly has resulted in some understanding of the structure of the multiloop amplitudes in the theory of closed, oriented boson strings in terms of the analytic geometry of modulus space.[1] So far, we do not know the expression for the amplitudes in the supersymmetry case. The partition functions (the p-loop vacuum diagrams) can be represented as integrals over the spaces M_p of the moduli of Riemann surfaces of type p with analytic measures $d\mu_{SS}^{(p)}(y)$ and $d\mu_{HS}^{(p)}(y)$ (y represents holomorphic coordinates on M_p):

$$\int_{M_p} (\det \operatorname{Im}\tau)^{-5}\,|d\mu_{SS}^{(p)}(y)|^2 \quad \text{for a superstring,}$$
$$\int_{M_p} (\det \operatorname{Im}\tau)^{-5}\,d\mu_{SS}^{(p)}\,\overline{d\mu_{HS}^{(p)}} \quad \text{for a heterotic string.} \tag{1}$$

If these theories do in fact have a ten-dimensional supersymmetry at the quantum level, the vacuum loops in them must vanish; more precisely, the identity $d\mu_{SS}^{(p)}(y)\equiv 0$ must hold. In this letter we propose a hypothesis regarding the structure of $d\mu_{SS}^{(p)}(y)$ which leads to the satisfaction of this identity.

2. A partition function for a superstring can be represented as an integral in a space of supermoduli with a definite measure. The supermodulus space can be represented by, for example, a factor in the modular group Sp (p, Z) of the direct product of $2^{p-1}(2^p+1)$ copies of the Teichmüller space $\widetilde{M}_p$ on a $(2p-2)$-dimensional plane space of "odd moduli."[1] The Teichmüller space $\widetilde{M}_p$ is the space of moduli of Riemann surfaces with the system of basis cuts noted above. M_p is a factor of $\widetilde{M}_p$ in the modulus group. The $2^{p-1}(2^p+1)$ copies of $\widetilde{M}_p$ are distinguished by the choice of even spinor structures.

The measure on the supermodulus space is determined by the ratio of the superdeterminants of the operators $\bar{\partial}$, which act on the superfields $\hat{x}(\xi)$ and $\hat{e}(\xi)$ (ten-dimensional coordinates of the string plus fermions and supertetrads). Expressions like (1) are found after an integration over odd moduli. Although the supermodulus space itself is connected, after an integration of this type a discrete sum over $2^{p-1}(2^p+1)$ copies of the ordinary modulus space M_p arises

$$d\mu_{SS}^{(p)}(y) \sim \sum_e C_e(y)(\det{}_e\bar\partial_{1/2})^5(y)\,[(\det{}'_e\bar\partial_{3/2})(y)]^{-1}\,. \tag{2}$$

We have singled out here only the contributions that depend on the θ-characteristics: the boundary conditions on the half-integer differentials. The weights $C_e(y)$ in this sum, which arise after the integration over odd moduli, are certain functions on M_p. They can be determined from the dependence of the superdeterminants on the odd moduli or in a different way, from the modulus properties of $d\mu_{SS}^{(p)}(y)$. In Sec. 4 we will use some slightly different arguments to determine them. In the calculation of the amplitudes, the weights $C_e(y)$ depend on the amplitude under consideration.

3. The dependence of $\det_e\bar\partial_{n+1/2}$ on the θ-characteristics e is determined primarily by the corresponding θ-constant $\theta[e]$: $\det_e\bar\partial_{n+1/2}(y) \sim \theta[e](y)$ ($\theta[e]$ depends on the matrix of periods τ, which is an analytic function of y; see Refs. 1 and 2 regarding this proportionality and Refs. 1–3 regarding the definition and properties of the θ-constants). This is the complete dependence, however, only for $n = 0$:

$$\det{}_e\bar\partial_{1/2} \sim \theta[e] \text{ (more precisely } [\det\bar\partial_0]^{1/2}(\det{}_e\bar\partial_{1/2}(y)) = \theta[e](y)^{\ 4}\text{)}\,. \tag{3}$$

If and only if there are holomorphic 1/2-differentials with the given boundary conditions on the Riemann surface does $\theta[e](y)$ vanish, and the number of differentials is equal to the order of the zero of $\theta[e](y)$. For at least the even characteristics (which are all that we are interested in) we can write

$$\det{}'_e\bar\partial_{3/2}(y) \sim \Phi_e^{-1}\,\theta[e](y),$$

$$\text{(more precisely } [\det\bar\partial_0]^{1/2}\det{}'_e\bar\partial_{3/2}(y) = \Phi_e^{-1}\,\theta[e](y)^{\ 4}\text{)}, \tag{4}$$

where $\det'_e \neq 0$ for any y, so that the function $\Phi_e(y)$ must completely cancel all the zeros of $\theta[e](y)$ and must not vanish when $\theta[e]\neq 0$. On the other hand, it must depend on the boundary conditions in an unambiguous way, in contrast with $\theta[e]$, which acquires phase factors under modulus transformations that do not change e. As $\Phi_e(y)$ we could choose the expression $\Phi_e = \det[\zeta_\alpha(P_1)...\zeta_\alpha(P_{p-1})\zeta'_\alpha(P_1)...\zeta'_\alpha(P_{p-1})]$. Here the $\zeta_1,...,\zeta_{2p-2}$ are holomorphic 3/2-differentials on a Riemann surface with boundary conditions corresponding to the given characteristic e, and the $P_1,..., P_{p-1}$ are the positions of the double zeros of the Priem holomorphic 1-differential $\nu^2 = \theta_i\omega_i$ (the ω_i are canonical holomorphic differentials; $\int_{a_i}\omega_j = \delta_{ij}$, $\int_{b_i}\omega_j = \tau_{ij}$). In the local coordinates ξ near P_μ we have $\zeta = [\zeta(P_\mu) + \zeta'(P_\mu)\xi + 0(\xi^2)](d\xi)^{3/2}$. If the condition $\theta[e] = 0$ holds for the Riemann surface (for the given y), the meaning is that there are holomorphic 1/2-differentials on it with given boundary conditions $\psi_1,..., \psi_k$, whose number k is equal to the order of the zero of $\theta[e]$ [see Eq. (3)]. The number of holomorphic 3/2-differentials $\zeta_1,..., \zeta_{2p-2}$ then includes $\nu^2\psi_1,..., \nu^2\psi_k$, each of which has double zeros at all points, P_μ so that $\det[\zeta_\alpha(P_\mu)\zeta'_\alpha(P_\mu)]$ has a zero of order k, and the right side of (4) vanishes nowhere. In contrast, if the determinant has a zero of order k, then some k holomorphic 3/2-differentials ζ_α have double zeros at all points P_μ. Working from them to ζ_α, we construct k holomorphic 1/2-differentials $\psi = 3\nu^{-2}$, whose existence is equivalent, by virtue of (3), to the situation that $\theta[e]$ has a k-fold zero.

Knizhnik recently gave a proof[4] of equations of the type in (3) and (4) on the basis of conditions imposed by conformal invariance and anomalies. These equations are given in parentheses in (3) and (4); also important for the anomalies are contributions which do not depend on the characteristics. For example, the anomaly in the product $\det^{1/2}\bar{\partial}_0 \det_e \bar{\partial}_{3/2}$ calculated in the metric $|v|^4 = \theta_{,i}\bar{\theta}_{,j}\omega_i\bar{\omega}_j$ on the Riemann surface is taken into account exactly by the P_μ dependence on the right side of (4).

4. By virtue of relations (2)–(4) we have $d\mu_{SS}^{(p)}(y) \sim \Sigma_e C_e(y)\Phi_e(y)\theta[e]^4(y)$. Our hypothesis is

$$C_e(y) \sim \epsilon_e[\Phi_e(y)]^{-1}, \tag{5}$$

where $\epsilon_e = \pm 1$ are certain phase factors (more on them below). Expression (5), like (1), of course contains factors which depend on y but not on the characteristic e. If this hypothesis is correct, the vanishing of $d\mu_{SS}^{(p)}(y)$ is ensured simply by the Riemann identity

$$\sum_e \epsilon_e \theta[e]^4(y) \equiv 0 \tag{6}$$

and depends on nothing else; in particular, it does not depend on the choice of metric on the Riemann surface. In the opposite cases, there would have to be an identity relation between the θ-constants and the residues of the holomorphic differentials, but this would hardly be possible. We also note that expression (5) is trivially valid for the case $p = 1$, in which we have $\Phi_e = 1$.

It should also be noted that our arguments are based exclusively on the properties of the system of zeros (divisors) of the θ-constants and are therefore simpler than a possible approach to the determination of $C_e(y)$ based on a study of the modulus properties of $d\mu_{SS}^{(p)}(y)$. In particular, the modulus properties of the residues of holomorphic differentials are quite complicated. Even in the boson case, all that has been determined in the way of modulus properties are expressions for $p \leqslant 4$ (Ref. 1b), while the expressions which follow from the properties of the divisors are known for all p (Ref. 1c) (but they may use superfluous information on the parametrization of the Riemann surfaces).

5. Let us prove identity (6). At $p \geqslant 2$, there are many independent Riemann relations among the fourth powers of the even θ-constants. (The number of these relations is $(4^p - 1)/3$, so that only $(2^p + 1)(2^{p-1} + 1)/3$ of the total number of $2^{p-1}(2^p + 1)\theta[e]^4$ are linearly independent.) An identity in which all of the even θ-constants enter on an equal footing should apparently appear in (6); i.e., all the factors ϵ_e are purely phase factors: $|\epsilon_e| = 1$. Unfortunately, there is no *a priori* reason for the existence of such a representation of the Riemann identities, and it must be specially derived. In fact, if we write the characteristic as

$$e = \begin{bmatrix} \delta_1 \dots \delta_p \\ \epsilon_1 \dots \epsilon_p \end{bmatrix}$$

we have

$$\sum_e (-1)^{\delta_1 + \epsilon_1} \theta[e]^4 \equiv 0. \tag{7}$$

The other identities are found from this one by means of modulus transformations [in practice, it is sufficient to use transformations from the factor group $Sp(p,Z)/Sp(p,2Z)$; the Riemann identities are invariant under transformations specified by simplectic matrices with even coefficients]. Identity (7) is a trivial consequence of the well-known relation[3]

$$\theta[{}^{\delta}_{\epsilon}]^2(\tau) = \sum_{\alpha} \theta[{}^{\alpha}_{0}](2\tau)\theta\begin{bmatrix}\alpha+\delta\\2\epsilon\end{bmatrix}(2\tau) = (-1)^{\epsilon\delta}\sum_{\alpha}(-1)^{\alpha\epsilon}\theta[{}^{\alpha}_{0}](2\tau)\theta[{}^{\alpha+\delta}_{0}](2\tau)\ , \tag{8}$$

which expresses $\theta(\tau)$ with an arbitrary characteristic in terms of θ with the characteristics $[{}^{\delta}_{0}]$, but the latter depend on the double parameter 2τ. [Here τ is a symmetric $p\times p$ matrix with a positive definite imaginary part: an element of a Siegel space. The equations for the string amplitudes contain τ's which are matrices of periods on Riemann surfaces of type p (Ref. 1).] To derive (7), it is sufficient to substitute into it four squared expressions (8) with

$$\begin{bmatrix}\delta_1\delta\\ \epsilon_1\epsilon\end{bmatrix} = \begin{bmatrix}0\,\delta\\ 0\,\epsilon\end{bmatrix};\begin{bmatrix}0\,\delta\\ 1\,\epsilon\end{bmatrix};\begin{bmatrix}1\,\delta\\ 0\,\epsilon\end{bmatrix};\begin{bmatrix}1\,\delta\\ 1\,\epsilon\end{bmatrix}$$

Examining similar terms in the sums over α, we easily see that we have an identical zero. Relation (8) itself is found from the definition

$$\theta\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}^2(\tau) \equiv \sum_{m\in Z^p} \exp i\pi\left[\left(m+\frac{\delta}{2}\right)\tau\left(m+\frac{\delta}{2}\right)+\left(m+\frac{\delta}{2}\right)\epsilon\right]$$
$$\times \sum_{n\in Z^p}\exp i\pi\left[\left(n+\frac{\delta}{2}\right)\tau\left(n+\frac{\delta}{2}\right)+\left(n+\frac{\delta}{2}\right)\epsilon\right]$$

by switching to a summation over the variables $\mu_i = 2^{-1}(m_i+n_i)$ and $v_i = 2^{-1}(m_i-n_i)$, which simultaneously take on integer or half-integer values. Here we need to use the summation formula

$$\left(\sum_{u_1 v_1\in Z} + \sum_{u_1 v_1\in Z+\frac{1}{2}}\right)\cdots\left(\sum_{u_p v_p\in Z} + \sum_{u_p v_p\in Z+\frac{1}{2}}\right) = \sum_{\alpha\in\{0,1\}^p}\ \sum_{u,v\in Z^p+\frac{\alpha}{2}}\ .$$

[1)]We are indebted to A. S. Shvarts for a discussion of this formulation. Regarding the Teichmüller space and its relationship with modulus space, we refer the reader to Ref. 1 and the bibliography there.

[1]a) A. A. Belavin and V. G. Knizhnik, Phys. Lett. **168B**, 201 (1986). b) A. A. Belavin, V. G. Knizhnik, and A. M. Perelomov, Pis'ma Zh. Eksp. Teor. Fiz. **43**, 319 (1986) [JETP Lett. **43**, 411 (1986)]; Preprint ITEP-59, 1986. c) Yu. I. Manin, Pis'ma Zh. Eksp. Teor. Fiz. **43**, 161 (1986) [JETP Lett. **43**, 204 (1986)]; A. Morozov, Preprint ITEP-88, 102, 1986.

[2]L. Alvarez-Gaumé, G. Moore, and C. Vafa, Harvard Preprint, HUTP-86/A017.

[3]D. Mumford, Tata Lectures on Theta, Birkhauser, 1983; J. D. Fay, *Theta Functions on Riemann Surfaces* (Springer, 1973) (LNM, No. 352); G. Clemens, Mosaic Theory of Complex Curves (Russ. transl. Mir, Moscow, 1984).

[4]V. G. Knizhnik, Europhys. Lett. (to appear, 1986).

Translated by Dave Parsons