Quantization of the mass of a black hole in string theory

Ya. I. Kogan
Institute of Theoretical and Experimental Physics

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A possible quantization of the mass of a black hole in accordance with $M \sim M_p \sqrt{n}$ in string theory is discussed. This quantization is based on the condition for a consistent description of the motion of a test string in the external field of a black hole. A possible link with the Hawking effect is examined.

Superstring theory[1,2] appears to be a candiate for the role of a unified theory of all interactions, including quantum gravitation. At the present level of development of string theory, however, it is not possible to directly study the quantum properties of nonperturbative objects such as black holes. Furthermore, it is not even clear whether black holes exist in a string theory. A systematic description of such entities would involve constructing coherent stationary states of string gravitons. The expectation value of the metric of these gravitons would be equal to the metric of the black hole. At present, we have no such construction, and the only way to study the matter is by the method of an external field,[3] in which one examines a string in a given external field. In other words, the string is a test object. The requirement of conformal invariance of the corresponding two-dimensional field theory describing the motion of a test string[4] or, equivalently, the vanishing of the β function, imposes definite restrictions on the external field, which are essentially equations of motion for the corresponding fields.[3] In this letter we examine certain restrictions on the metric of a black hole which follow from another principle: the principle of the separation of the left-hand and right-hand modes of a string in a given external gravitational field, which leads to a quantization of the Schwarzschild radius and thus of the mass of the black hole. For simplicity, we will discuss a boson string, which in this case leads to the same results as for a superstring.

1. We consider a string in the external field of a black hole. The critical dimensionality of the space in this case is $D = 26$ ($= 10$ for fermion strings). After compactification, the vacuum state is $M^4 \times K^{D-4}$, where K is a compact manifold (or orbifold). In the case of a black hole, M^4 must be replaced by a space with the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{r_g}{r}\right) dt^2 + \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 , \tag{1}$$

where r_g is the Schwarzschild radius (the gravitational radius).

Since the manifold K remains constant, we can restrict the analysis to a four-dimensional sector (in all the equations below, we will omit the contributions of the other $D - 4$ dimensions, for brevity). The action of a closed string in an external field is

$$S = \frac{1}{4\pi\alpha'} \int d\sigma d\tau g_{\mu\nu}(x) \partial_a x^\mu \partial_a x^\nu , \qquad a = 1, 2 \quad \mu, \nu = 1, .., 4 , \tag{2}$$

and for a positive action we need to carry out an analytic continuation into the Euclidean region, $t \rightarrow it$. In other words, instead of (1), we need to examine

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 + \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 . \tag{1'}$$

At $r \geqslant r_g$, this metric describes a smooth manifold with the topology $R^2 \times S^2$ (or $S^2 \times S^2$ for compactification of infinity); hence t is an angular variable with a period of $4\pi r_g$ (Ref. 5). That this is true can be seen quite easily by making the substitution $r = r_g + y^2$ in the limit $r \rightarrow r_g$. The metric then becomes $ds^2 \sim dy^2 + (y/2r_g)^2 dt^2 + r_g^2 d\Omega^2 + 0(y)$, from which we see that the singularity at $y = 0$ is a coordinate singularity of the polar coordinate system at the center of the plane.

Let us examine the equations of motion which follow from (2):

$$\partial_a \partial_a x^\mu + \Gamma^\mu_{\nu\rho}(x) \partial_a x^\nu \partial_a x^\rho = 0 \tag{3}$$

or

$$\partial^2_{z\bar{z}} x^\mu + \Gamma^\mu_{\nu\rho}(x) \partial_z x^\nu \partial_{\bar{z}} x^\rho = 0,$$

with $z = \sigma + \tau$, $\bar{z} = \sigma - \tau$ (for the analytic continuation, $z = \sigma + i\tau$). It is easy to see that (3) allows solutions of the form

$$x_L = x(z) = x(\sigma + \tau), \qquad x_R = x(\bar{z}) = x(\sigma - \tau). \tag{4}$$

For quantization of the string we would need to expand solutions(4) in oscillators (see Ref. 1, for example); this expansion is

$$x_R^\mu(\sigma,\tau) = x^\mu + 2\alpha' p_R^\mu(\sigma-\tau) + \frac{i}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a_n}{n} e^{2in(\sigma-\tau)} \tag{5}$$

with the commutation relations

$$[x^\mu, p_R^\nu] = i\delta^{\mu\nu}, \qquad [a_n^\mu, a_m^\nu] = 2\alpha' n\,\delta_{n+m,0}. \tag{6}$$

Since in our case $x^0 = t$ is an angular variable, the spectrum of the operator p_R^0 is discrete, and its eigenvalues are

$$p_R^0 = n/2r_g, \qquad n = 0, \pm 1, \ldots \quad . \tag{7}$$

In addition, there is one more condition on p_R^0, because we are considering closed strings. We therefore have $x^\mu(0,\tau) = x^\mu(\pi,\tau)$, but in the case $\mu = 0$ we could generalize the closure condition, $x^0(0,\tau) = x^0(\pi,\tau) + 4\pi m r_g$, $m = 0, \pm 1,\ldots$, from which we find

$$p_R^0 = 2mr_g/\alpha'. \tag{8}$$

There seems to be no point in considering $|m| > 1$, since these states are unstable (they decay to states with $|m| = 1$, which cannot decay only to states with $m = 0$). Accordingly, an analysis of the states in the Euclidean sector may prove incorrect, so that we are left with $|m| = 1$. From the two conditions (7) and (8) we immediately find a condition for the quantization of the gravitational radius:

$$r_g^2 = \frac{\alpha'}{4} n, \qquad n = 0, 1, \ldots \quad . \tag{9}$$

This condition is equivalent to the condition that the mass of the black hole is quantized:

$$M_{B.H.} = \frac{\alpha'^{1/2}}{4G_N}\sqrt{n}, \qquad n = 0, 1, \ldots \quad . \tag{9'}$$

2. The arguments above indicate that in a systematic quantum theory the mass of black holes must be quantized. More-rigorous arguments could be found in an approach based on a σ-model analysis of a string in an external field, i.e., a study of a two-dimensional quantum field theory of the type

$$\int Dx^\mu(\sigma,\tau)\, e^{-\frac{1}{4\pi\alpha'}\int d\sigma\, d\tau\, g_{\mu\nu}(x)\,\partial_a x^\mu\, \partial_a x^\nu + \ldots} \tag{10}$$

in the case of a boson string and its supergeneralization for a superstring.

The arguments above could be reformulated in different terms as the following hypothesis.

The supersymmetric σ model, which describes the structure (or heterotic structure) in an external field with metric (1′), is conformally invariant only for values of r_g which satisfy quantization condition (9). In other words, the β-function $\beta(r_g)$ vanishes exactly at these values of r_g. In general, in these models the β-function is multicharged[6] because of a change in the form of the external metric upon renormalizations, but in the case of the Schwarzschild metric the form of the metric remains the same, and only the coupling constant r_g is renormalized. These results are a reflection of the Birkhoff theorem regarding spherically symmetric solutions of Einstein's equations.

The model discussed here has instanton solutions, since the fields take on values from the manifold $S^2 \times S^2$. In Kruskal coordinates, the action is

$$U = (r/r_g - 1)^{1/2}\, e^{\frac{r+it}{2r_g}}, \quad r \geqslant r_g$$

$$\mathcal{L} = \frac{4r_g^3}{r}\, e^{-r/r_g}\, \partial_z U \partial_{\bar z} U^* + r^2(|\partial_z \theta|^2 + \sin^2\theta\, |\partial_z \varphi|^2). \tag{11}$$

$$4r_g^3/r \exp(-r/r_g) = f(|U|^2).$$

The equations of motion for U are

$$f(|U|^2)\partial^2_{z\bar z} U + f'(|U|^2) U^* \partial_{\bar z} U \partial_z U = 0.$$

From these equations we see that the classical solutions are holomorphic functions, as in the O(3) model.[7]

3. In conclusion we examine some consequences of a quantized mass of a black hole. The difference between the masses of two adjacent levels $n+1$ and n is $M_{n+1} - M_n = \alpha'/32G_N^2 M_{B.H.}^{-1}$, so that in transitions between these levels quanta will be emitted with a characteristic energy $\epsilon = M_{n+1} - M_n$. The dependence of this energy on $M_{B.H.}$ is analogous to the dependence of the Hawking evaporation temperature, $T_H = 1/8\pi G_N M_{B.H.}^{-1}$ which is proportional to the characteristic energy of the thermal quanta. There is thus the hope that it will be possible to describe the evaporation of black holes in string theory. Here it will be necessary to develop a formalism in which the external field would be different in "in" and "out" states. These topics will be discussed further in a detailed paper. We simply note here that a corresponding expression for the spectrum could be found by applying the Bohr-Sommerfeld quantization rules to the Euclidean action of a black hole,[5] $S_{B.H.} = 4\pi G_N M_{B.H.}^2 = 2\pi n$. We find, in general, a different proportionality coefficient in front of $n^{1/2}$, but if complete agreement with (9) is required, we could relate α' and G_N.

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