Rate of particle production in gravitational fields

Ya. B. Zel'dovich

Institute of Applied Mathematics, USSR Academy of Sciences

A. A. Starobinskiĭ

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences
(Submitted August 15, 1977)
Pis'ma Zh. Eksp. Teor. Fiz. **26**, No. 5, 373–377 (5 September 1977)

The local rate of production of massless particles is calculated in a weakly anisotropic homogeneous cosmological model, and also in a weak inhomogeneous gravitational field, in which the condition $q^i q_i \geq 0$ is satisfied for the wave vectors of the nonzero Fourier components in second-order perturbation theory. The rate turns out to be proportional to the local values of the invariants of the curvature tensor.

PACS numbers: 98.80.–k

It is well known that particle-antiparticle pairs can be produced in an alternating gravitational field, including those with zero rest mass. Different aspects of these effects were considered in a large number of papers. In particular, we have previously calculated[1] the mean value of the energy-momentum tensor (EMT) of the produced scalar particles in the case when the external classical gravitational field is described by a homogeneous anisotropic

 0021-3640/77/2605-0252$00.60

Bianchi type-I metric. This mean value turns out to be a complicated nonlocal causal functional of the space-time metrics. It includes a contribution from the real produced particles and the polarization of the vacuum of the quantum field, and in the general case the splitting of the unified mean value of the EMT into an EMT of the real particles and an EMT that describes the polarization of the vacuum is not general-covariant and unique. The reason is the ambiguity in the definition of the concept of a particle as a field quantum (in contrast to the operator of the EMT of the field) in Riemannian space-time.

If, however, the space-time is flat at $t=\pm\infty$, then we can uniquely define the in- and out-vacuums and calculate the number of real particles produced from the in-vacuum in all of space during the entire time interval $-\infty<t<\infty$. In this paper we point out that in some cases it is possible to attain the local rate of production of massless particles per unit volume and per unit time, and this rate depends only on the local values of the invariants of the Riemann tensor.

As the first example we consider a weakly anisotropic homogeneous cosmological Bianchi type-one model, and calculate the local rate of production of massless conformally covariant particle $g^{-1/2}[d(g^{-1/2}n)/dt]$ to second order in the anisotropy (i.e., to first order in the small ratio $C_{iklm}C^{iklm}/R_{iklm}R^{iklm}$, where C_{iklm} is the Weyl conformal tensor). The space-time metric takes the form[1)]

$$ds^2 = dt^2 - a^2(t)[(1+h_1(t))dx_1^2 + (1+h_2(t))dx_2^2 + (1+h_3(t))dx_3^2], \tag{1}$$

where $|h_\alpha|\ll 1$, $\sum_{\alpha=1}^{3}h_\alpha=0$. Introducing, in accordance with[1], the variable $\eta=\int dt/a(t)$ and, in the case of production of scalar particles, the field function $\chi=\phi/a$, we obtain for the space-like Fourier component (to first order in h_α):

$$\chi_k'' + k^2\chi_k = V_k(\eta)\,\chi_k, \tag{2}$$

where

$$V_k(\eta) = \sum_{a=1}^{3} h_a(\eta)\,k_a^2, \quad k^2 = \sum_{a=1}^{3} k_a^2; \quad \mathbf{k}=\{k_a\}$$

is a constant wave vector. If the quantum field was in a vacuum state at $t=-\infty(\eta=-\infty)$, then $\chi_k=\exp(-ik\eta)$ as $\eta\to-\infty$. Then (2) can be represented in the form of an integral equation:

$$\chi_k(\eta) = \frac{1}{k}\int_{-\infty}^{\eta} \sin k(\eta-\eta_1)V_k(\eta_1)\,\chi_k(\eta_1)d\eta_1 + e^{-ik\eta}. \tag{3}$$

As $t\to+\infty$ $(\eta\to+\infty)$, when $h_\alpha\to 0$, we have $\chi_k(\eta)=\alpha_k\exp(-ik\eta)+\beta_k\exp(ik\eta)$, where $|\alpha_k|^2-|\beta_k|^2=1$. Then

$$\alpha_k = 1 + \frac{i}{2k}\int_{-\infty}^{\infty} e^{ik\eta}\,V_k(\eta)\chi_k(\eta)\,d\eta, \tag{4}$$

$$\beta_k = -\frac{i}{2k}\int_{-\infty}^{\infty} e^{-ik\eta}\,V_k(\eta)\,\chi_k(\eta)\,d\eta. \tag{5}$$

The total density of the real produced particles is connected with β_k as $\eta\to+\infty$ by the formula:

$$n = (2\pi)^{-3}a^{-3}\int d^3k\,|\beta_k|^2. \tag{6}$$

Solving (3) by iteration and substituting $\chi_k(\eta)$ in (5), we find that to second order in h_α we have as $\eta \to +\infty$:

$$n = \frac{1}{1920\pi a^3} \int_{-\infty}^{\infty} d\eta \sum_{\alpha=1}^{3} (h_\alpha'')^2 = \frac{1}{960\pi a^3} \int_{-\infty}^{\infty} d\eta\, a^4 C_{iklm} C^{iklm} \tag{7}$$

(the prime denotes differentiation with respect to η). Therefore the local rate of production of massless scalar particles in the metric (1) is[2)]:

$$\frac{1}{\sqrt{-g}} \frac{d}{dt} (\sqrt{-g}\, n) = \frac{1}{960\pi} C_{iklm} C^{iklm} \geqslant 0 \ . \tag{8}$$

It can be shown by the same method that an analogous formula holds also for the production of photons and neutrinos. The only difference is that the numerical coefficient in the right-hand side of (8) is $1/320\pi$ for neutrinos and $1/80\pi$ for photons.

We consider now the production of gravitons in the Friedmann isotropic model ($h_\alpha = 0$), when furthermore the ratio $R^2/R_{iklm}R^{iklm}$ is small (R is the scalar curvature). Using the equation for gravitational waves,[2] which reduces to the form:

$$\chi_k'' + \left(k^2 - \frac{a''}{a} \right) \chi_k = 0,$$

and the methods described above, we arrive at the following results:

$$\frac{1}{\sqrt{-g}} \frac{d}{dt} (\sqrt{-g}\, n_g) = \frac{R^2}{288\pi} \geqslant 0. \tag{9}$$

At $R = 0$ and $C_{iklm} = 0$, no gravitons are produced—this result was noted by Grishchuk.[3]

We note, however, that in the next higher order in the small parameter

$$\frac{C_{iklm}C^{iklm}}{R_{iklm}R^{iklm}}$$

(or $R^2/R_{iklm}R^{iklm}$) the density of the produced particles $n(\eta = +\infty)$ is not expressed in terms of a single integral of local quantities with respect to η, and therefore the local formulas (8) and (9) cannot be generalized to include higher orders of perturbation theory.

We proceed to the case of a weak inhomogeneous gravitational field. Let the space-time metric be of the form $g_{ik} = \eta_{ik} + h_{ik}$, where η_{ik} is the Minkowski metric and $|h_{ik}| \ll 1$. Then the total probability of pair production, to second order in h, is

$$W = \langle 0 | S^{(1)} + S^{(1)} | 0 \rangle = -2\,\mathrm{Re}[\langle 0 | S^{(2)} | 0 \rangle], \tag{10}$$

where $S^{(1)}$ and $S^{(2)}$ are the corresponding terms of the S-matrix expansion in powers of h. Calculating S by perturbation theory, in analogy with the procedure used in[4], we obtain:

$$W = \frac{a}{8\pi} \frac{1}{(2\pi)^4} \int d^4q [R_{ik}(q)(R^{ik}(q))^* - \frac{1}{3} | R(q)|^2] \theta(q^0)\theta(q^i q_i), \quad (11)$$

where the coefficient α is equal to 1/60 in the case of massless conformal scalar particles, 1/20 in the case of neutrinos, and 1/5 in the case of photons (for details of the calculations see[5]). On account of the factor $\theta(q^i q_i)$, the integral in (11) cannot be transformed in the general case into a single integral with respect to d^4x. If, however, the condition $q^i q_i \geq 0$ is satisfied at $R_{ik}(q) \neq 0$, then the θ function of $q^i q_i$ in (11) can be omitted, and then (taking into account the known fact that the invariants $C_{iklm}C^{iklm}$ and $2(R_{ik}R^{ik} - \frac{1}{3}R^2)$ differ by a total derivative) W reduces to the form $W = \int w(x^l) d^4x$

$$w(x^i) = \frac{a}{32\pi} C_{iklm} C^{iklm} \quad (12)$$

which leads again to (8) in the homogeneous case.

At the present time, during the Friedmann stage of the evolution of the universe, no neutrinos or photons are produced, and the rate of production of gravitons, as follows from (9), is vanishingly small, $\sim 10^{-106}$ cm^{-3} sec^{-1}. Near the singularity we have $(-g)^{-1/2}[d(-g^{-1/2}n)/dt] \sim t^{-4}$, which yields $(d\epsilon/dt) \sim t^{-5}$, in agreement with the results of[1]. Formula (8) is valid also in the case of particles with mass, provided that $\hbar^{-4}m^4 \ll | R_{iklm}R^{iklm}|$.

1)We chose a system of units in which the speed of light is equal to unity.
2)The result (8) can be easily obtained from the formulas given in the Appendix 1 of[1].

1Ya. B. Zel'dovich and A. A. Starobinskiĭ, Zh. Eksp. Teor. Fiz. **61**, 2161 (1971) [Sov. Phys. JETP **34**, 1159 (1972)].
2E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **16**, 587 (1946).
3L. P. Grishchuk, Zh. Eksp. Teor. Fiz. **67**, 825 (1974) [Sov. Phys. JETP **40**, 409 (1975)].
4R. U. Sexl and H. K. Urbantke, Phys. Rev. **179**, 1247 (1969).
5A. A. Starobinskii, Candidate's Dissertation, Inst. Theor. Phys. USSR Acad. Sci., Chernogolovka, 1975.