Rotating ^{3}He-A

G. E. Volovik and N. B. Kopnin

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences
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It is shown that rotation of a vessel with ^{3}He-A produces a periodic structure consisting of unique vortices and having no singularities in the superfluid-component velocity.

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When a vessel with ^{3}He-A is rotated, the normal component of the liquid should rotate as a unit (i.e., $\mathbf{v}_n = \omega \times \mathbf{r}$, where ω is the angular velocity of the rotation), and the field of the superfluid velocity $\mathbf{v}_s$ and of the anisotropy vector $\mathbf{l}$ should give a minimum of the functional $\tilde{F} = F - \mathbf{L}\omega$, where F and $\mathbf{L}$ are the total free energy and the total angular momentum of the liquid. By using the hydrodynamic expression given in[1] for the superfluid parts of the free energy and the angular momentum, we reduce the functional $\tilde{F}$ to the form

$$\tilde{F} = \int d^3\mathbf{r} \Big\{ \frac{1}{2} \rho_s (\mathbf{v}_s - [\tilde{\omega}, \mathbf{r}])^2 - \frac{1}{2} \rho_0 (\mathbf{l}, \mathbf{v}_s - [\tilde{\omega}, \mathbf{r}])^2 + C(\mathbf{v}_s - [\tilde{\omega}, \mathbf{r}], \operatorname{rot}\mathbf{l}) - C_0(\mathbf{l}, \mathbf{v}_s - [\tilde{\omega}, \mathbf{r}])(\mathbf{l}, \operatorname{rot}\mathbf{l}) + \frac{1}{2} K_1 (\operatorname{div}\mathbf{l})^2 + \frac{1}{2} K_2 (\mathbf{l}, \operatorname{rot}\mathbf{l})^2 + \frac{1}{2} K_3 [\mathbf{l}, \operatorname{rot}\mathbf{l}]^2 \Big\} - \frac{1}{2} \rho \int d^3\mathbf{r} \, [\tilde{\omega}, \mathbf{r}]^2. \tag{1}$$

Here ρ is the total density of the liquid; the quantities ρ_0, mC, mC_0 and $m^2K_{1,2,3}$ are of the order of ρ_s. In a normal liquid, a similar functional is minimized by the field of the velocities $\mathbf{v} = \omega \times \mathbf{r}$. In He II, such a field $\mathbf{v}_s$ is impossible by virtue of the vanishing of curl $\mathbf{v}_s$ in the interior of the liquid. The minimum is ensured here by the lattice of vortex filaments, which at sufficiently large $\omega \gg (mR^2)^{-1}$, where R is the radius of the vessel, imitate on the average the field $\omega \times \mathbf{r}$. The superfluid velocity near each filament has a singularity $v_s = 1/mr$ ($\hbar = 1$), yielding an energy gain in comparison with the rigid-body rotation

$$\Delta\tilde{F} = (\pi \rho_s \omega R^2/2m) \ln (r_c/\xi), \tag{2}$$

where r_c is the cell dimension, and ξ is the radius of the core of the vortex. In ^{3}He-A such a structure is also possible in principle, and consists of vortex filaments with singularities at $\mathbf{v}_s$. For these vortices we have $\mathbf{l} \parallel \omega$, and the energy loss is determined as before by (2), where now ξ is the coherence radius.

We shall show that in ^{3}He-A there is realized a structure of unique vortices, which also imitates on the average the field of the superfluid velocities $\omega \times \mathbf{r}$, but is energywise less favored than (2), since its superfluid velocity has no singularities, and the energy loss therefore does not contain a logarithmic contribution: $\Delta\tilde{F} \sim \rho_s \omega R^2/m$. This is due to the fact that curl $\mathbf{v}_s \neq 0$, but satisfies the condition[1]

$$(\mathrm{rot}\ \mathbf{v}_s)_i = \frac{1}{4m} e_{ijk}\left(\mathbf{l}, \left[\frac{\partial \mathbf{l}}{\partial x_j} \times \frac{\partial \mathbf{l}}{\partial x_k}\right]\right). \tag{3}$$

This expression is a consequence of the fact that the order parameter in ^{3}He-A is specified by a triad of unit vectors $\mathbf{\Delta}'$, $\mathbf{\Delta}''$, $\mathbf{l}$, the orientation of which relative to the coordinate system with z axis along ω can be described, for example, by three Euler angles α, β, γ. We have thus for $\mathbf{l}$ and $\mathbf{v}_s$

$$\mathbf{l} = (\sin\beta\cos\alpha;\ \sin\beta\sin\alpha;\ \cos\beta),$$

$$\mathbf{v}_s = \frac{1}{2m}[(1-\cos\beta)\nabla\alpha + \nabla\Phi]; \quad \Phi = -(\alpha+\gamma). \tag{4}$$

The angles α and β specify the position of $\mathbf{l}$ on the unit sphere. By virtue of the homogeneity along the z axis, all the quantities depend only on $\mathbf{r} = (x, y)$, and $\mathbf{v}_s$ lies in the (x, y) plane.

The minimum of $\widetilde{F}$ will be attained when $\mathbf{v}_s$ is close to $\omega\times\mathbf{r}$. To this end it is necessary that the integral of curl $\mathbf{v}_s$ over any macroscopic region S such that $S \gg (m\omega)^{-1}$ is almost completely canceled out by the term $2\omega S$. Retaining only the principal terms in $m\omega S \gg 1$, we obtain (see (3))

$$\frac{1}{2}\int_S e_{ijk}\left(\mathbf{l}, \left[\frac{\partial \mathbf{l}}{\partial x_j} \times \frac{\partial \mathbf{l}}{\partial x_k}\right]\right) dS_i = 4 m\omega S \gg 1. \tag{5}$$

It can be assumed, with the same accuracy, that the left-hand side of (5) is equal to $4\pi N$, where $N \gg 1$ is an integer, and the influence of the vessel boundary can be neglected. The quantity $4\pi N$ is equal to the area swept on the unit spere by the vector $\mathbf{l}$ when $\mathbf{r}$ runs over S, since the left-hand side of (5) is equal to $\int \sin\beta d\beta d\alpha$. It is clear from (1) and (5) that the terms with derivatives of $\mathbf{l}$ in F are minimal when the derivatives $|\partial l_i/\partial x_k| \sim (m\omega)^{1/2}$ along any direction. It follows therefore that when $\mathbf{r}$ traces the boundary of the region S the vector $\mathbf{l}$ traces on the unit sphere a closed line of length $O(N^{1/2}) \ll N$. Accurate to the principal terms in N, this means that the boundary of the region swept on the sphere contracts to a point, i.e., at the same accuracy the vector $\mathbf{l}$ runs N times through all the values on the sphere (with allowance for the direction of its circuiting motion). We now break up S into cells S_α, $\Sigma_\alpha S_\alpha = S$, such that when $\mathbf{r}$ moves over S_α the vector $\mathbf{l}$ travels over the sphere once. It is clear that the characteristic cell dimensions r_c in any direction should be of the order of $(m\omega)^{-1/2}$.

One should expect the cells to be periodic, each cell having an area $S_0 = \pi/m\omega$ It is easily seen that $\mathbf{v}_s - \omega\times\mathbf{r}$ will also be periodic and will be of the order of ωr_c. A cell in a two-dimensional periodic lattice is topologically equivalent to the surface of a torus. Since the field $\mathbf{l}$ produces a mapping of degree unity of the torus on a sphere, the cell can be chosen in such a way that its boundary is mapped on one point of the sphere. Consequently, $\mathbf{l}$ is constant on the boundary of such a cell. From the symmetry of the problem it can be assumed that on the boundary we have $\beta = 0$ or $\beta = \pi$ (we assume for the sake of argument $\beta = 0$). It follows from (3) that the circulation of $\mathbf{v}_s$ over the boundary is equal to $2\pi/m$, and the angles α and γ change by -2π on going around the center of the cell (the point where $\beta = \pi$). The circulation of $\mathbf{v}_s$ about the lines $\beta = $ const is equal to $(\pi/m)(1+\cos\beta)$ and decreases from $2\pi/m$ to zero on moving from the boundary to the center. It appears that the cells will have the most symmetrical form, i.e., quadratic or hexagonal.

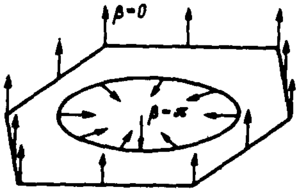

FIG. 1. One of the possible fields of the vector **l** in the hexagonal cell.

The condition of the smoothness of Δ', Δ'', and **l** on the boundary calls for $\partial\beta/\partial\mathbf{n} = 0$ (**n** is the normal to the boundary) and for $\nabla\Phi$ to be continuous. The velocity v_s is then automatically continuous. We note that if we put $\beta = 0$ everywhere in (4), we obtain a lattice of vortex lines with two circulation quanta, since Φ changes by 4π on going around the center of the cell. Thus, the investigated structure can be obtained from a vortex lattice with two circulation quanta, by going over continuously from a constant field $\mathbf{l} = \hat{\mathbf{z}}$ into a field such as the one shown, for example, in Fig. 1. The singularity on the line then vanishes. The topological feasibility of such a transition has been noted in[2]. The considered field **l** in the cell recalls the structure discussed in[1,3].

The periodic lattice can be observed, for example, by NMR methods, since the NMR frequency depends on the orientation of **l** relative to the magnetic field. At large rotation velocities, when $r_c \sim \xi$, the superfluidity will be suppressed. This phenomenon is analogous to H_{c2} in type-II superconductors. The critical angular velocity of the rotation is, however, very large: $\omega_c \sim (m\xi^2)^{-1} \sim 10^5\ \mathrm{sec}^{-1}$.

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