Integrability of a classical *XY* chain

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The stationary Landau-Lifshitz integral is found for a discrete *XY* chain. The problem is therefore integrable. The solutions are found explicitly. Their energy is found. A phase diagram is plotted.

Among dynamic problems in which a stochastic behavior can arise,[1] the problem of the stationary states of a chain of $2D$ classical spins (the *XY* model) has recently attracted particular interest. It was subjected to numerical analysis in Refs. 2 and 3, where contradictory interpretations were offered: Belobrov *et al.*[2] believe that the problem is integrable, while Thompson *et al.*[3] assert that a complete chaos arises in the solutions.

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This problem has an exact solution, which we report here.

The Hamiltonian of the XY model is

$$\mathcal{H} = - \sum_n (\mathbf{S}_n, \hat{J}\mathbf{S}_{n+1}), \tag{1}$$

where $\mathbf{S}_n = (S_n^x, S_n^y) = (\cos\theta_n, \sin\theta_n)$ is a vector of unit length, and the exchange-constant matrix $\hat{J}$ can be chosen without any loss of generality to be of the form

$$\hat{J} = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}, \quad J > 1.$$

The stationary states of Hamiltonian (1) are described by the Landau-Lifshitz equation

$$[\mathbf{S}_n, \hat{J}(\mathbf{S}_{n-1} + \mathbf{S}_{n+1})] = 0, \tag{2}$$

which is a nonlinear second-order difference equation, which may be thought of as a mapping of the spin space into itself.

According to Eq. (2), the $(n+1)$-st spin vector can be chosen in two ways: either

$$\mathbf{S}_{n+1} = -\mathbf{S}_{n-1} \tag{3}$$

or in such a way that the expression

$$W = (\mathbf{S}_n, \hat{J}^{-1}\mathbf{S}_{n+1}) \tag{4}$$

is independent of n.

Here we will consider only the latter ("regular") case. The general solution is obviously constructed from regular pieces of various lengths, which differ in the sign of W [operation (3) changes the sign of W].

For regular solutions, expression (4) is therefore an integral of Eq. (2).

In the phase plane of the variables

$$u = \theta_n + \theta_{n+1}, \quad v = \theta_n - \theta_{n+1}$$

relation (4) specifies a family of phase curves

$$(1 + J^{-1})\cos v + (1 - J^{-1})\cos u = 2W, \tag{5}$$

which differ in the value of the parameter W. It is sufficient to study the phase diagram in the square $0 \leqslant u, v \leqslant \pi$ (Fig. 1; $J = 4$). The phase curves with $W > 0$ and $W < 0$ are symmetric with respect to each other around the center of this square. In terms of spins, this symmetry reduces to the mapping

$$\mathbf{S}_n |_{W<0} = (-1)^n \mathbf{S}_n |_{W>0}. \tag{6}$$

The parameter region $0 \leqslant W < J^{-1}$ corresponds to the solution

$$S_n^x = \cos n(qn + \varphi_0; k), \quad S_n^y = \sin n(qn + \varphi_0; k), \tag{7}$$

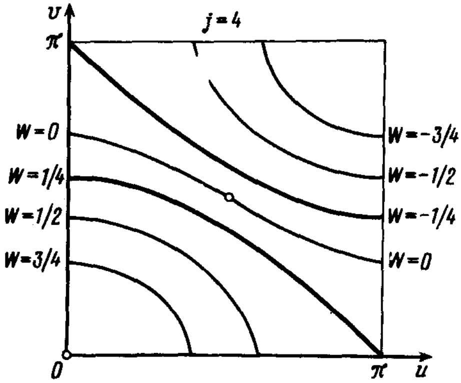

FIG. 1.

where q and k satisfy the equations

$$\mathrm{dn}(q;k) = J^{-1}; \qquad k^2 = \frac{1-J^{-2}}{1-W^2} . \tag{8}$$

This solution describes a structure with a complete spin flipping; its period is generally incommensurate with the lattice constant.

At values $J^{-1} < W < 1$, the elliptic modulus satisfies $k^2 > 1$, and the corresponding solution describes spins which are oscillating around a "hard" axis with an angular amplitude $\theta_1 = \arcsin(1/k)$. We can also write explicit expressions for this case:

$$S_n^x = \mathrm{dn}(qn + \varphi_0; k), \qquad S_n^y = k_1 \sin n(qn + \varphi_0; k_1), \tag{7a}$$

$$\cos n(q; k_1) = J^{-1}; \qquad k_1 = 1/k. \tag{8a}$$

The separatrix between these solutions ($W = J^{-1}$; i.e., $k = k_1 = 1$) corresponds to a solution of the Landau-Lifshitz domain-wall type[4]:

$$S_n^x = 1/\cos n(qn + \varphi_0), \qquad S_n^y = \tanh(qn + \varphi_0), \cosh q = J. \tag{9}$$

The energy of these solutions,

$$\mathcal{E} = -\sum_{n=0}^{N-1} (\mathbf{S}_n, \hat{J}\mathbf{S}_{n+1}),$$

can be calculated exactly (cf. Ref. 5). For solution (7), for example, we have

$$\mathcal{E} = -N\left[\cos nq + \frac{(J-J^{-1})E(q)}{k^2 \sin nq}\right] + \frac{J-J^{-1}}{k^2 \sin nq}[E(qN+\varphi_0) - E(\varphi_0)], \tag{10}$$

where $E(u)$ is the elliptic integral of the second kind.

In the limit $k \to 1$ ($W \to J^{-1}$), we find the energy of a piece of the domain wall consisting of N spins:

$$\mathscr{E}_{dw} = \sqrt{J^2 - 1}\,[\tanh(qN + \varphi_0) - \tanh\varphi_0] . \tag{11}$$

We have thus demonstrated the complete integrability of the problem of the stationary solutions of an XY chain. The integral is the quantity W in (4). The presence of this integral is responsible for the circumstance that all of the subsequent mappings of the spins conform to a fixed phase curve. This curve is filled densely everywhere for arbitrary values of the two initial spins.[2]

In the case in which the period is commensurate with the lattice constant, however, the curve consists of a finite number of points. For solution (7), the condition of commensurability means $qm_1 = 4Km_2$ ($m_{1,2}$ are integers), where K is the complete elliptic integral of the first kind corresponding to modulus k. Thompson *et al.*[3] took the appearance of an incommensurability as the onset of chaos in this solution, but this interpretation does not conform to reality.

The property of the integrability of the stationary solutions of spin chains is not an exclusive property of $2D$ spins. It can be shown that for classical XYZ chain the integral W is accompanied by the appearance of yet another (complementary) integral of the Landau-Lifshitz equation.[6]

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