Helfrich deformation in smectic A liquid crystals

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The onset of an instability of a smectic A liquid crystal in a magnetic field directed parallel to the layers is analyzed. Regions in which the layers are periodically modulated can form in the system. The modulation amplitude and the characteristic parameters describing the motion of the boundaries of these regions are found.

A Helfrich deformation, consisting of the formation of a periodic bending of the smectic layers, can arise in a homeotropically oriented smectic A liquid crystal with a positive diamagnetic anisotropy χ_a in a magnetic field H directed parallel to the smectic planes.[1] The reason for the onset of this modulated structure is an instability of smectics with respect to a perturbation of their macroscopic homogeneity. The standard way for describing this effect is to find the characteristic period of the final structure and the strength of the critical field by studying the free energy averaged over the volume of the sample. When that approach is taken, it is not possible to draw any conclusions about the modulation amplitude, its relationship with the initial perturbation, or the time evolution of the establishment of the final structure. It thus seems worthwhile to study the nonlinear dynamics of the process, which will give us at least qualitative answers to these questions.

To describe the nonlinear dynamics of a smectic A liquid crystal, we follow Refs. 2 and 3, using the variable $W(\mathbf{r},t)$ to describe the layered structure ($W=$ const specifies the position of a layer of molecules).

The free energy density of the smectic A liquid crystal is written as an expansion in the gradients of this function; the leading terms of this expansion are

$$F_W = \frac{B}{8}[l^2(\nabla W)^2 - 1]^2 + \frac{Kl^2}{2}(\nabla^2 W)^2 . \tag{1}$$

Here l is the equilibrium distance between the smectic layers, and B and K are elastic moduli. At equilibrium we would have $W_0 = z/l$, and this quantity would describe a system of layers perpendicular to the z axis.

When there is a deviation from equilibrium, it is convenient to write the variable W as

$$W = \frac{z+u}{l} ,$$

where u is the displacement of the layers along the z axis. We assume a geometry such that the smectic liquid crystal lies between two bounding surfaces $z=0$ and $z=d$, and the magnetic field is directed along the x axis. This field is uniform throughout the sample. The problem thus becomes effectively two-dimensional. The magnetic field is incorporated by adding to (1) a term $-\frac{1}{2}\chi_a(\mathbf{nH})^2$, where the director $\mathbf{n}$ is defined by

$$\mathbf{n} = \nabla W / |\nabla W|.$$

From the hydrodynamic equations[3] for a smectic A liquid crystal, if we ignore heat conduction and leakage, we find a nonlinear equation of motion for the displacement u, which we write as follows in terms of the dimensionless variables $Z = (\pi/d)z$, $X = (\pi\nu/\lambda d)^{1/2}x$, $T = (8\nu\pi K/\lambda d\eta)t$, and $u = 2^{3/2}\nu^{1/2}\lambda\varphi$ [here ρ is the density, η is the corresponding viscosity coefficient, $\lambda = (K/B)^{1/2}$, $\nu = (H - H_c)/H_c$, and $H_c = (2K\pi/\chi_a\lambda d)^{1/2}$]:

$$-\frac{2^6\rho K}{\eta^2}\varphi_{TT} + 4\varphi_{XX} + \nu^{-2}\varphi_{ZZ} - \frac{2H^2}{\nu H_c^2}\varphi_{XX} - \varphi_{XXXX} + 2^{5/2}\nu^{-1/2}\varphi_X\varphi_{XZ}$$
$$+ \varphi_{XX}(12\nu\varphi_X^2 + 2^{3/2}\nu^{-1/2}\varphi_Z) = 0. \quad (2)$$

Analysis of the linearized version of Eq. (2) shows that under the condition $2^{1/2}H_c > H \gtrsim H_c$ an initial state $\varphi = 0$ becomes unstable with respect to perturbations $\delta\varphi \sim \exp(ikX + iqZ + \omega T)$ with $q = 1$ and $k \sim k_c = \nu^{-1/2}$ (here $\omega \simeq 1$). These perturbations will grow; as a result, a region of a quasiperiodic structure with a wavelength of approximately $2\pi\nu^{1/2}$ should form. The nature of the structure and the dynamics of its formation are determined by nonlinear equation (2). When we are just slightly above the critical level, $\nu \ll 1$, the solution of Eq. (2) can be written

$$\varphi = \sin\eta[V(X,T)\exp(i\nu^{-1/2}X) + V^*(X,T)\exp(-i\nu^{-1/2}X)], \quad (3)$$

where V is a complex function which depends on X and T and which varies over scale dimensions $X,T \sim 1$.

Substitution of (3) into (2) leads to the following equation for the amplitude V:

$$\frac{\partial V}{\partial T} = \frac{\partial^2 V}{\partial X^2} + V - V|V|^2. \quad (4)$$

This is the "amplitude equation." It has been used by many authors to describe a long list of physical phenomena (Refs. 4–7 and the bibliographies there).

Analysis of Eq. (4) shows that its solution corresponding to a transition from an unstable state $V = 0$ to a stable, bounded, stationary state is real and represents a wave traveling at a velocity $c = 2$. The shape of the wavefront is determined by the solution of the ordinary differential equation

$$\frac{d^2V}{dy^2} + 2\frac{dV}{dy} + V - V^3 = 0.$$

This front describes a transition from the state $V = 0$ to a state $V = \pm 1$.

In terms of dimensional variables, the modulation amplitude of the periodic structure which forms is $u_0 = \lambda(2\nu)^{1/2}$; since $\lambda \sim l$, this amplitude is smaller than the distance between layers (i.e., defects of the layered structure do not form) if we are just slightly above the critical level. Both in the region of the periodic structure and near the wavefront we have $u_z \sim \nu^{1/2}\lambda/d \ll 1$, $u_x \sim \nu(\lambda/d)^{1/2} \ll 1$ at $\lambda < d$. Consequently, if the sample is not too thin, the conditions under which Eq. (2) was derived are satisfied.

Working from the linearized version of Eq. (4), we can find the scale time for the formation of the front; in dimensionless variables this time is

$$T \sim -\ln|Z_0(0)|,$$

where $z_p(0)$ is the Fourier transform of $Z(X,T)$ at $T=0$.

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