Divergence of the perturbation-theory series and pseudoparticles

L. N. Lipatov

B. P. Konstantinov Institute of Nuclear Physics, USSR Academy of Sciences
(Submitted July 23, 1976; resubmitted November 3, 1976)
Pis'ma Zh. Eksp. Teor. Fiz. **25**, No. 2, 116–119 (20 January 1977)

It is shown that the asymptotic form of the coefficients of expansion of the Gell-Mann–Low function in scalar theories of the form $H_{\rm int}=\phi^4/4!$ is determined by spherically symmetrical solutions of the classical equations in Euclidean 4-dimensional space and by the quantum fluctuations near these solutions.

PACS numbers: 11.10.Ef, 11.10.Gh, 11.10.Jj

1. Nonlinear classical field equations with exact solutions have attracted interest in recent years.[1] The hypothesis has been advanced that the presence of finite-energy stationary solutions (solitons) of these equations offers evidence in favor of the appearance of stable states in the spectrum of the corresponding quantum problem.[2] Nonstationary solutions with finite action were also obtained for the Yang—Mills model in Euclidean four-dimensional space (the so-called pseudoparticles).[3] The question of the corollaries that can be deduced for the quantum problem from the existence of nonstationary solutions is still open at present. In this paper it is shown, using scalar theories with dimensionless coupling constants as an example, that the degree of divergence of the perturbation-theory series is determined by spherically-symmetrical solutions of the corresponding classical equations in four-dimensional space at a negative sign of the coupling constant. An exact estimate of various quantities in higher orders of perturbation theory is also of practical interest. Thus, for example, in quantum electrodynamics, the accuracy with which the anomalous magnetic moment is measured will necessitate in the nearest future calculations of a tremendous number of eight-order perturbation-theory diagrams. One can hope that the methods developed in the present paper can be generalized to the case of quantum electrodynamics.

2. We consider a scalar theory in D-dimensional Euclidean space-time with a Hamiltonian

$$H=\int d^Dx\left[\frac{(\partial_\nu\phi)^2}{2}+g_\mu\frac{\phi^n}{n!}\right]+\int d^Dx\,\mathcal{H}'(\phi,g_\mu), \tag{1a}$$

and assume for the renormalizability of the theory

$$D = 2n/(n-2), \qquad n = 4,6\,\ldots, \qquad g_\mu > 0. \tag{1b}$$

The constant g_μ is the invariant charge

$$g(p^2/\mu^2, g_\mu) \equiv g_\mu \Gamma_n(p^2/\mu^2, g_\mu)\, d^{n/2}(p^2/\mu^2, g_\mu) \tag{2}$$

at the normalization point $p^2 = p_\mu^2 = \mu^2$, where $\Gamma_n = d = 1$. The vertex function $\Gamma_n(p^2/\mu^2, g_\mu)$ is calculated at the symmetrical space-like point $p_i^2 = p^2 > 0, p_i p_j|_{i\neq j} = -(p^2/n-1)$, and $\Delta(p) = p^{-2} d(p^2/\mu^2, g_\mu)$ is the Green's function of the scalar particle. The counterterm $\mathcal{H}'$ in (1a) serves to eliminate ultraviolet divergences in subdiagrams with $m \leq n$ points:

$$\mathcal{H}' = -\frac{1}{2}\Delta^\circ(0) g_\mu \frac{\phi^{n-2}(x)}{(n-2)!} + g_\mu^2 \frac{\phi^n(x)}{\left(\frac{n}{2}!\right)^3} \frac{1}{2}\left(\frac{1}{4}\Gamma\left(\frac{D}{2}-1\right)\pi^{-(D/2)}\right)^{n/2}$$

$$\times \int \frac{d^D x}{x^D} \exp\left(i\frac{n x p_\mu}{\sqrt{n-1}}\right) + \ldots . \tag{3}$$

In this paper we obtain the asymptotic behavior, as $K\to\infty$, of the expansion coefficients $C_k(n)$ of the Gell-Mann—Low function

$$\psi(g(p^2/\mu^2, g_\mu)) = \frac{\partial g(p^2/\mu^2, g_\mu)}{\partial \ln p^2/\mu^2} = \sum_{k=2}^{\infty} (-g(p^2/\mu^2, g_\mu))^k C_k(n). \tag{4}$$

In the class of theories (1a) and (1b), the models of physical interest are those with $n=4$, $D=4$ [4] and $n=6$, $D=3$. [5] In the limit as $n\to\infty$, the coefficients $C_k(n)$ were obtained in my preceding paper in arbitrary order of perturbation-theory. [6]

3. In high orders, the main contribution to the invariant charge (2) is made by the corrections to the vertex part Γ_n. To calculate them it is necessary to find the kth order of the expansion of the n-point Green's function

$$G_n^{(k)} = I_0^{-1}\int \prod_x d\phi(x) \oint \frac{dg_\mu}{g_\mu^{k+1} 2\pi i} \prod_{r=1}^{n} \phi(x_r) e^{-H}, \qquad I_0 = \int \prod_x d\phi(x) e^{-H_0}. \tag{5}$$

It is easy to verify that as $k\to\infty$ the functional integral (5) contains a set of saddle-point values for $\phi(x)$ and g_μ:

$$\tilde{\phi}(x) = \pm\sqrt{k}\left(\frac{y}{y^2 + (x-x_0)^2}\right)^{\frac{D}{2}-1} \frac{\sqrt{2}}{D-2}\pi^{-\frac{D}{4}}\left\{\frac{\Gamma(D)}{\Gamma\left(\frac{D}{2}\right)}\right\}^{1/2},$$

$$-\tilde{g}_\mu = k^{-\frac{n-2}{2}} n!\left[\frac{8\pi}{(n-2)^2}\right]^{n/2}\left\{\frac{\Gamma(D/2)}{\Gamma(D)}\right\}^{\frac{n-2}{2}} \tag{6}$$

From the point of view of the Feymann diagram technique, this fact means the existence of a saddle-point distribution density $\tilde{p}(x) = -\tilde{g}_\mu(\tilde{\phi}^n/n!)$ for the interaction point in the kth order of perturbation theory.

The summation in (5) over the contributions of the saddle points with different centers x_0 and scales y is carried out in standard fashion. The quantum fluctuations near the saddle-point values (6) are calculated by diagonalizing the quadratic form in the small deviations $\Delta\phi$ and Δg_μ, which results from the expansion of H. The counterterm $\mathcal{H}'(\tilde{\phi}, \tilde{g}_\mu)$ eliminates in this case the produced ultraviolet divergences.

After removing the external Green's functions from $G_n^{(k)}$ and changing over to the vertex function Γ_n in momentum space, we obtain, using formulas (2) and (4), the following expression for the asymptotic values of the coefficients $C_k(n)$:

$$\lim_{k\to\infty} C_k(n) = (-\tilde{g}_\mu)^{-k} e^{k(1-\frac{n}{2})} k^{\frac{n+D}{2}} \left\{ 2^{1+\frac{D}{2}} \pi^{\frac{1}{4}(D-1)} \left[\Gamma\left(\frac{D}{2}\right)\right]^{-1} \left[\Gamma\left(\frac{D+1}{2}\right)\right]^{\frac{1}{2}} \right\}^n$$

$$\times (4\pi)^{-\frac{1}{2}} \left\{ \frac{D(D+2)}{4\pi(D+1)} \right\}^{\frac{D+1}{2}} \exp\left\{ -\frac{1}{2} \sum_{m=2}^{\infty} (D+2m-1) \frac{\Gamma(D+m-1)}{\Gamma(D)\Gamma(m+1)} \right.$$

$$\left. \times \left[\ln(1-\beta_m) + \beta_m \right] - \frac{D^2/2}{D-2} \right\}, \quad \beta_m = \frac{D(D+2)}{(D+2m)(D+2m-2)}$$

at $n \geq 6$ and

$$\tilde{C}_k \equiv \lim_{k\to\infty} C_k(4) = \left(\frac{k}{e\,16\pi^2}\right)^k k^4 \exp(3c_E - 8) 2^{19/2} 3^4 5^{-5/2} \pi \int_0^\infty dy\, y^6 [K_1(y)]^4$$

$$\times \exp\left\{ -\frac{1}{2} \sum_{m=2}^{\infty} \frac{2m+3}{a_m} \left[\ln(1-a_m) + a_m + \frac{a_m^2}{2} \right] \right\} \approx \left(\frac{k}{e\,16\pi^2}\right)^k 2.75 k^4, \tag{7}$$

$$c_E \approx 0.5772, \quad K_1(y) = y^{-1} \int_0^\infty \frac{dx}{(x^2+1)^{3/2}} \cos(xy), \quad a_m = \frac{C}{(m+1)(m+2)}$$

for the case of the theory with $H_{\text{int}} = g_\mu \int d^4x (\phi^4/4!)$.

It can be verified that $\lim_{n\to\infty} \lim_{k\to\infty} C_k(n)$ (see (5)) coincides with the asymptotic form, as $k \to \infty$, of the coefficients $C_k(n)$ obtained in the preceding paper for the case of large n. [6]

How soon does the asyptotic regime set in? Let us compare the first three coefficients of the Gell-Mann—Low function, obtained by direct calculations of the Feyman diagrams, [4] with their values obtained from the asymptotic formula (6) ($C_k(4) = (16\pi^2)^{-k+1} A_k$):

$$A_2 = 3/2, \quad A_3 = 17/6, \quad A_4 = 19.2$$
$$\tilde{A}_2 = 0.15, \quad \tilde{A}_3 = 1.89, \quad \tilde{A}_4 = 20.9 \tag{8}$$

Thus, the asymptotic regime is reached sooner, giving grounds for hoping that the correction terms $\sim 1/K$ in (6), which can also be calculated, have an anomalously small coefficient. This fact raises the hope of the possibility of independently verifying the result[6], that an ultraviolet stable points exists in the scalar theory. The next task is to calculate the Gell-Mann—Low function in the Yang—Mills theory and in quantum electrodynamics.

The author thanks A. A. Belyavin, A. P. Bukhvostov, V. N. Gribov, A. M. Polyakov, A. S. Shvarts, and D. V. Shirkov for useful discussions.

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