New Lorentz-invariant system with exact multisoliton solutions

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A Lorentz-invariant equation is derived for a scalar complex field in two-dimensional space-time. Exact two-soliton solutions are obtained for this equation and can be generalized in trivial fashion to include the case of N solitons.

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The recently increased interest in classical localized solutions of field-theory equations is due, in particular, to the possibility of constructing new particle models.[1] In the more thoroughly investigated two-dimensional case, besides fully integrable systems with trivial dynamics,[2] there exist systems in which soliton interaction is inelastic.[3,4] Such systems, in our opinion, represent a more adequate basis for the construction of realistic models of extended particles. The study of such systems, however, is made difficult by the lack of regular methods for their analytic description. On the other hand, the value of exactly solvable fully integrable equations lies, in particular, in the possibility of their use to describe, in first-order approximation, the solitons and "inelastic" systems that are "close" to them. The viability of this concept was demonstrated in[5]. In the present paper we propose a system with a symmetry group, for which exact multisoliton solutions are obtained, and "proximity" to this system can be used to explain the properties of the interaction of solitons of the nonlinear complex Klein-Gordon equation.[4]

We consider the theory of a complex scalar field in two-dimensional space-time with a Lagrangian

$$L = \frac{|\psi_\mu|^2}{1 - \lambda^2 |\psi|^2} - m^2 |\psi|^2 \qquad (|\psi_\mu|^2 = \partial^\mu \psi \partial_\mu \psi^* ;$$

$$\mu = 0, 1; \quad x_1 = x; \quad x_0 = t; \quad \psi_0 = \psi_t; \quad \psi_1 = \psi_x; \quad g_{00} = -g_{11} = 1). \tag{1}$$

The equation of motion

$$\partial_\mu^2 \psi + \lambda^2 \psi^* \frac{\psi_\mu^2}{1-\lambda^2|\psi|^2} + m^2\psi(1-\lambda^2|\psi|^2) = 0 \tag{2}$$

has a single-soliton solution, which we shall find convenient to write in the form

$$\psi = \frac{1}{\lambda}\,\frac{e^z}{1+ae^{z+z^*}} \,. \tag{3}$$

Here

$$z = z' + iz'' = k_\mu (x - x^{(o)})^\mu \tag{4}$$

is a Lorentz-invariant complex variable, k_μ is a complex space-like vector in two-dimensional pseudo-Euclidean space-time:

$$k_\mu^2 = k_o^2 - k_1^2 = -m^2\,; \tag{5}$$

$a=[(k+k^*)_\mu^2]^{-1}$; $x_\mu^{(0)}$ is an arbitrary constant vector, and $x_\mu^{(0)}=0$ fixes the position of the soliton at the origin of the pseudo-Euclidean space. The complex vector k_μ is characterized by four parameters, of which only two are independent by virtue of (5); the following choice of k is convenient:

$$k_o = m\,\mathrm{sh}(\beta \pm i a); \qquad k_1 = m\,\mathrm{ch}(\beta \pm i a). \tag{6}$$

By putting $\cosh\beta=\gamma$, $\gamma=(1-v^2)^{-1/2}$ and $\cos\alpha=A$, $|A|\le 1$, we can rewrite (3) in the following more conventional form:

$$\psi = \frac{1}{\lambda}\,\mathrm{sech}\, z' \exp(\pm iz'')$$

$$= \lambda^{-1} A\,\mathrm{sech}[Am\gamma(x-vt)+\delta_1]\exp\{\pm i[\sqrt{1-A^2}\,m\gamma(vx-t)+\delta_2]\}\,;$$

$$\delta_1 - \delta_2 = \ln 2A\,. \tag{7}$$

Obviously, v has the meaning of the soliton velocity while A characterizes the amplitude and "width" of the soliton and the frequency of the natural oscillations.

We note that the single-soliton solution of the nonlinear Klein—Gordon equation

$$\partial_\mu^2\psi + m^2(1-2\lambda^2|\psi|^2)\psi = 0 \tag{8}$$

is of exactly the same form, and in this sense Eqs. (2) and (8) are "close."

The integrals of the motion—the energy E, the momentum $\mathbf{p}$, and the charge Q—for the solution (7) are given by

$$E = \gamma M; \qquad \mathbf{p} = \gamma v M; \qquad M = 4Am\lambda^{-2}; \quad E^2 = \mathbf{p}^2 + M^2; \tag{9}$$

$$Q = \pm 4 \arcsin A = \pm \bar{Q} \tag{10}$$

(the sign of the charge corresponds to the sign of the imaginary unit in (4) and

(7)). Thus, the solutions of (7) are objects with properties of relativistic particles of mass M, charge $\pm Q$, and characteristic dimension $L \sim 2/Am\gamma$ (which is subject to Lorentz contraction as it moves) and having internal degrees of freedom described by the parameter A (or α).

It can be shown that the solutions (7) are stable in the entire range of variation of the parameter A. The reason, in particular, is that M and Q [Eqs. (9), (10)] have no extrema as functions of A (the analogous quantities for (8) are extremal at the point $A=\sqrt{2}/2$ at which the stability is lost).[4,6]

The numerical experiments have shown that the solitons (7) experience fully elastic scattering in collisions. This fact, and also that in the real limit Eq. (2) reduces via the transformation $\phi = 2 \arcsin\psi$ to the "sine-Gordon" equation for the function ϕ, gives grounds for assuming that Eq. (2) has exact multi-soliton solutions. We succeeded in finding the exact two-soliton solution by the Hirota method.[7] A trivial generalization of this solution is the N-soliton formula.

We seek the solution of (2) in the form

$$\psi = g(x,t)/f(x,t), \tag{11}$$

where f is a real function.

The solutions of the rather cumbersome equation for g and f are, in particular, solutions of a much simpler system of equations

$$f\partial_\mu^2 f - (\partial_\mu f)^2 + |g|^2 = 0$$

$$(\partial_\mu^2 g + g)(f^2 - |g|^2) - 2f\partial_\mu f\,\partial_\mu g + g^*(\partial_\mu g)^2 + g(\partial_\mu f)^2 = 0 \tag{12}$$

(the constants m and λ were eliminated by the similarity transformation $mx_\mu \to x_\mu$, $\lambda\psi \to \psi$). We introduce the quantities

$$a(i,j^*) = -[(k_{(i)} + k^*_{(j)})^2_\mu]^{-1}; \quad a(i,j) = -(k_{(i)} - k_{(j)})^2_\mu; \quad a(i^*,j) = [a(i,j^*)]^*;$$

$$a(i^*,j^*) = [a(i,j)]^*; \quad a(i,j,k^*) = a(i,j)a(i,k^*)a(j,k^*);$$

$$a(i,j,m^*,n^*) = a(i,j)a(i,m^*)a(i,n^*)a(j,m^*)a(j,n^*)a(m^*,n^*), \tag{13}$$

where $k_{(i)\mu}$ are defined in (6), and $k_{(i)\mu} \neq k_{(j)\mu}$ if $i \neq j$. It can be directly verified that g and f, defined by the formulas

$$g = e^{z_1} + e^{z_2} + a(1,2,1^*)e^{z_1+z_2+z_1^*} + a(1,2,2^*)e^{z_1+z_2+z_2^*} \tag{14}$$

$$f = 1 + a(1,1^*)e^{z_1+z_2^*} + a(1,2^*)e^{z_1+z_2^*} + a(2,1^*)e^{z_2+z_1^*} + a(2,2^*)e^{z_2+z_2^*}$$

$$+ a(1,2,1^*,2^*)e^{z_1+z_2+z_1^*+z_2^*}, \tag{15}$$

where $z_i = k(i)_\mu (x - x_i^{(0)})^\mu$, satisfy the system (12).

Consequently, formulas (11) and (13)—(15) yield for Eq. (2) a solution that describes the scattering of two solitons. The parameters that determine the ith solution are "hidden" in the variable z_i. The generalization of (13)—(15) to the case $N>2$ is quite obvious.

In the limit as $t\to\pm\infty$ the solution breaks up into a sum of two solutions of the type (7), with phase shifts that can be easily calculated.

Assuming the parameters β_i in (6) to be complex, we obtain an expression describing the bound state of two solitons. We present an explicit form of the expression describing the bound state of solitons with equal but opposite charges in the rest system ($\alpha_1=-\alpha_2=\alpha$; $\beta_1=-\beta_2=i\xi$):

$$\psi=\cos\xi\cos\alpha\sin(\alpha+\xi)\,e^{\sin\alpha\sin\xi x}\,\tilde{g}/\tilde{f};$$

$$\tilde{g}=e^{-\cos\alpha\cos\xi x-\delta_1}\cos[\sin(\alpha+\xi)t]-e^{\cos\alpha\cos\xi x+\delta_1}\cos[\sin(\alpha-\xi)t];$$

$$\tilde{f}=\cos^2\xi\cos^2(\sin\xi\cos\alpha\ t)-\cos^2\alpha\sin^2(\cos\xi\sin\alpha t) \tag{16}$$

$$+\operatorname{sh}(\cos\alpha\cos\xi x+\delta_2)(\cos^2\xi-\cos^2\alpha);$$

$$\delta_1=\ln\frac{\sin(\alpha+\xi)}{2\cos\alpha\cos\xi};\qquad \delta_2=\ln[\sqrt{\sin^2\alpha-\sin^2\xi}/2\cos\alpha\cos\xi].$$

It is remarkable that *in both* limiting cases $\alpha=0$ and $\xi=0$ the expression (16) takes the form $\psi=\sin\phi/2=\sin 2\arctan[\tan\theta\ \mathrm{sech}(\cos\theta x)\sin(\sin\theta t)]$ ($\theta=\xi$ or $\theta=\alpha$, respectively); ϕ is the "bion" of the "sine-Gordon" equation.[8]

Experience in the study of equations having exact multisoliton solutions shows that they have infinite sets of integrals of motions and probably all are fully integrable systems. One can hope the system described here to have similar properties.

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[1]I.S. Shapiro, Zh. Eksp. Teor. Fiz. **70**, 2050 (1976) [Sov. Phys. JETP **43**, 1069 (1976)]; "Extended systems in field theory," Phys. Rev. **23C** (1976).
[2]V.E. Zakharov, in: I.A. Kunina, Teoriya uprugikh sred s mikrostrukturoĭ (Theory of Elastic Media with Microstructures), Nauka, 1976.
[3]A.E. Kudryavtsev, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 178 (1975) [JETP Lett. **22**, 82 (1975)]; B.S. Getmanov, Pis'ma Zh. Eksp. Teor. Fiz. **24**, 323 (1976) [JETP Lett. **24**, 291 (1976)].
[4]B.S. Getmanov, JINR R2-10208, Dubna, 1976.
[5]B.S. Getmanov, JINR R2-9996, Dubna, 1976; Phys. Lett. (to be published).
[6]L.G. Zastavenko, Fiz. Metal. Metalloved, **29**, 430 (1965).
[7]R. Hirota, J. Phys. Soc. Jap. **33**, 1459 (1972).
[8]L.D. Faddeev and L.A. Takhtadzhyan, Dokl. Akad. Nauk SSSR **219**, 1334 (1974) [Sov. Phys. Dokl. **19**, 824 (1975)]; P.J. Caudrey, J.C. Eilbeck, and L.D. Gibbon, Nuovo Cimento **25B**, 497 (1975).