Regularization of gauge theories; non-Abelian anomalies

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A regularization of gauge theories is proposed. This regularization meets the necessary physical requirements and leads to values of the non-Abelian anomalies which are different from the generally accepted values.

The anomalous divergences of currents play a fundamental role in the physics of elementary particles. In the present letter we propose a regularization of the fermion fluctuations in gauge theories. This regularization leads to values [see (7) below] of the Weyl or non-Abelian anomalies which are different from the generally accepted values.[1-4] Our regularization corresponds to a regularization in energy, which is used in the Hamiltonian formalism.[5] The regularization methods which are explicitly relativistically invariant in the Feynman-diagram approach and also the regularization in the Hamiltonian formalism, which lead to unitary and relativistically invariant theories, are therefore fundamentally different. Since the anomaly in (7) below leads to a "less malignant" breaking of the gauge invariance of the system of an interacting gauge field and a right-hand "left-hand" Weyl field, there is the real hope that such a system can be quantized in a self-consistent way (see also Ref. 10).

1. In a $(3+1)$-dimensional Minkowski space we consider the right-hand Weyl field $\phi(x)$ in the external c-number gauge field, $A_\mu(x) = A^a_\mu(x)t^a$. The fermion part of the action is $S_\varphi = \int d^4x\varphi^+(u\nabla_0 + u\sigma^i\nabla_i)\varphi$. Everywhere below $\nabla_\mu = \partial_\mu + A_\mu$, σ^i are the Pauli matrices, t^a are anti-Hermitian generators of the Lie algebra of the gauge group, the Greek-letter indices are four-dimensional, and the Italic Latin indices are three-dimensional. We will construct a regularized transition amplitude of fermions, which we denote by $Z_+\{A_\mu\}$. We denote by $\{\varphi_N(x)\}$ the complete orthonormal set of solutions of the Weyl equation; i.e.,

$$(i\nabla_0 + i\sigma^i\nabla_i)\varphi_N = 0, \tag{1}$$

$$\sum_N \varphi_N(t,\mathbf{x})\overset{+}{\varphi}_N(t,\mathbf{y}) = \delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{2}$$

Everywhere, we are using $(t,\mathbf{x}) = (x)$. By virtue of completeness condition (2), arbitrary fields $\varphi(x)$ and $\varphi^+(x)$ can be expanded in the set of functions $\{\varphi_N(x)\}$ and $\{\varphi^+{}_N(x)\}$ with time-dependent coefficients $\{a_N(t),\bar{a}_N(t)\}$. The set of these coefficients may be regarded as a complete set of Grassmann degrees of freedom of the system. It is convenient to regularize the fermion transition amplitude in terms of these variables. To do this, it is sufficient to discard the ultraviolet (in the energy) tails in the expansions of the fields $\varphi(x)$ and $\varphi^+(x)$ in terms of the functions $\varphi_N(x)$ and $\varphi^+{}_N(x)$. Everywhere below, a prime on a summation or product over N means that these indices *do not* run over values in the ultraviolet tail. We define regularized Fermi fields as follows:

$$\varphi(x) = \sum_N{}' a_N(t)\varphi_N(x), \quad \overset{+}{\varphi}_N(x) = \sum_N{}' \bar{a}_N(t)\overset{+}{\varphi}_N(x) . \tag{3}$$

We partition the time into small segments of length $\epsilon \to 0: t_{\kappa+1} = t_\kappa + \epsilon$. According to the general rules, the coordinate variable $\{a_N(t_\kappa)\}$ are determined at the times t_κ, while the momentum variables $\{\bar{a}_N(t_\kappa + \epsilon/2)\}$ are determined at the times $t_\kappa + \epsilon/2$. The regularized fermion measure and transition amplitude are defined by

$$(D\bar{\psi}D\psi)_{P+} = \prod_\kappa \prod_N{}' \delta\bar{a}_N(t_\kappa + \epsilon/2)\, \delta a_N(t_\kappa), \tag{4}$$

$$Z_+\{A_\mu\} = \text{const} \int (D\bar{\psi}D\psi)_{P+} \exp iS_\varphi .$$

Transition amplitude (4) is unitary, since the Hamiltonian of the fermions is zero in terms of the variables $\{a_N, \bar{a}_N\}$. After the cutoff is removed, there is also relativistic invariance.

In precisely the same way we can define the regularized measure $(D\bar{\psi}D\psi)_P$ and the regularized transition amplitude $Z_-\{A_\mu\}$ of the left-hand Weyl field $\chi(x)$.

We denote by $(D\bar{\psi}D\psi)_P$ the regularized measure of a massless Dirac field $\psi(x)$ in an external c-number gauge field. By definition, we set

$$(D\bar{\psi}D\psi)_P = (D\bar{\psi}D\psi)_{P+} (D\bar{\psi}D\psi)_{P-} . \tag{5}$$

Since we have $S_\psi = S_\varphi + S_\chi$ for the action of a massless Dirac field, it follows from definitions (4) and (5) that

$$Z\{A_\mu\} = Z_+\{A_\mu\} Z_-\{A_\mu\}, \tag{6}$$

where $Z\{A_\mu\}$ is the regularized and P-even transition amplitude of the massless Dirac field.

2. To derive the anomalies I will use the method which I proposed[6] in 1978. This method can be summarized as follows: Upon a change in variables in a functional integral, one should take into account the change in the functional measure, which also gives rise to an anomaly in the corresponding Ward identity. This method was first used in Ref. 7 (see also Ref. 8). In our case, we should make the replacements $\varphi \to \varphi + v\varphi$, $\varphi^+ \to \varphi^+ - \varphi^+ v$, in the integral in (4), where the field $v(x) = v^a(x)t^a$ is an infinitesimal parameter of the transformation. Direct calculations lead to the expression

$$\nabla_\mu J_\varphi^{\mu a} = \frac{i}{32\pi^2} \epsilon^{\mu\nu\lambda\rho} \operatorname{tr} t^a F_{\mu\nu} F_{\lambda\rho} , \tag{7}$$

where $J_\varphi^{\mu a}$ $(J_\chi^{\mu a})$ is the current constructed from the right-hand (left-hand) Weyl field φ (or, respectively, χ).

Similar manipulations with the integral in (6) yield the following relations for the currents $J^{\mu a} = \bar{\psi} i t^a \gamma^\mu \psi$ and $J_5^{\mu a} = \bar{\psi} i t^a \gamma^\mu \gamma^5 \psi$: $\nabla_\mu J^{\mu a} = 0$ and also $\nabla_\mu (J^{\mu a} - J_\varphi^{\mu a} - J_\chi^{\mu a}) = \nabla_\mu (J_5^{\mu a} - J_\varphi^{\mu a} + J_\chi^{\mu a}) = 0$. The latter relations follow immediately from factorizations (5) and (6). The gauge invariance of amplitude (6) guarantees unitarity and renormalizability of the P-even theory.

3. The value which we have found for the anomaly in (7) contradicts the self-consistent Wess-Zumino equation,[2] since the quantity

$$\mathfrak{A}_+ \{v\} = \int d^4 x v^a \nabla_\mu \langle J_\varphi^{\mu a} \rangle$$

does not satisfy that equation. We will now show how to resolve this paradox.

We denote by L_v the generator of a gauge transformation in Minkowski space: $A_\mu (x)] = \nabla_\mu v(x)$. We have

$$L_v Z_+ \{A_\mu\} = -i \mathfrak{A}_+ \{v\} Z_+ \{A_\mu\}. \tag{8}$$

We postulate a new Lie algebra of the gauge group, after modifying the commutation relations by introducing a Schwinger term:

$$[L_u, L_v] = L_{[u, v]} - h \frac{1}{32\pi^2} \int d^4 x [u, v]^a \epsilon^{\mu\nu\lambda\rho} \operatorname{tr} t^a F_{\mu\nu} F_{\lambda\rho}. \tag{9}$$

It is easy to verify that commutation relations (9) satisfy the Jacobi identities. From Eqs. (8) and (9) we find a modified self-consistent Wess-Zumino equation:

$$L_u \mathfrak{A}_+ \{v\} - L_v \mathfrak{A}_+ \{u\} = \mathfrak{A}_+ \{[u, v]\} - h \frac{i}{32\pi^2} \int d^4 x [u, v]^a \epsilon^{\mu\nu\lambda\rho} \operatorname{tr} t^a F_{\mu\nu} F_{\lambda\rho}. \tag{10}$$

With $h = 1$, the quantity

$$\mathfrak{A}_+ \{v\}$$

constructed in accordance with (7) satisfies Eq. (10). The paradox is thus eliminated.

4. In the transformation to a Euclidean space (to calculate the partition function), one should use the ordinary temperature technique (Ch. 3 in Ref. 9), in which the regularization described here and the anomaly in (7) are realized in an obvious way.

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