“Phase transitions” in an array of parametric waves at the surface of an oscillating liquid

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Recent experiments by Aleksandrov *et al.* on water in a vessel with a vibrating bottom revealed that regular structures with hexagonal or square cells and an array of parallel crests are formed on the surface of the water under certain conditions. The present letter offers a theory for this phenomenon which is induced by a parametric resonance.

1. In vibrating vessels holding a liquid one sometimes observes a wave ripple with a regular structure. This phenomenon was apparently described first in 1831 by Michael Faraday.[2] In the experiments by Ezerskiĭ *et al.*[3] on a vibrating thin layer of silicone, a grid of waves of squares was observed, while three types of grids were observed by Aleksandrov *et al.*[1]: a hexagonal grid, a grid of squares, and a one-dimensonal grid (Fig. 1).

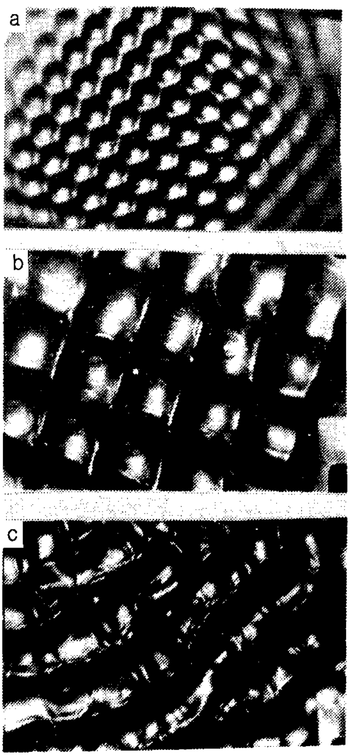

FIG. 1. Some representative motion-picture frames of wave lattices with (a) hexagonal, (b) square, and (c) ridge cells.

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This phenomenon is somewhat similar to a phase transition which occurs in a crystal, e.g., upon the conversion of graphite into diamond, with a change in lattice type. Another possibility is that the experiments of Ref. 1 could be thought of as a laboratory modeling of the "seaquakes" which sometimes accompany submarine earthquakes. According to eyewitness reports,[4] a grid of waves with lengths of 10–20 m and frequencies of 1 Hz arises for 10–60 s at the sea surface. There is the possibility that if the amplitudes are large, these events could also pose a danger to shipping. In particular, there were reports of the loss of ships in the open sea near the center of the Mexican earthquake in 1985.

We might also note that similar but steady-state structures arise in a dielectric or ferromagnetic liquid in a vertical electric field[5–7] or a vertical magnetic field[8] (see Refs. 9 and 10 for a corresponding theory).

2. We begin with an elementary analysis. Waves on water have a dispersion law $\omega^2 = kg$ (curve 1 in Fig. 2), or $\omega^2 = kg + k^3 \times (\sigma/\rho)$ when the surface tension σ is taken into account (curve 2 in Fig. 2). If the bottom is vibrating periodically in accordance with $\delta z = H_\theta(t) = H \cos \omega t$ at a velocity $v_0(t) = -\omega H \sin \omega t$ and with an acceleration $\delta g = -\omega^2 \cos \omega t$, this acceleration must be added to the acceleration due to gravity, so that the perturbation equation must be written

$$\ddot{h} + [k(g + \delta g) + (\sigma/\rho) k^3] h = 0, \quad (1)$$

where a dot means the partial derivative with respect to the time t. This Mathieu equation[11] can describe the growth of parametric waves with the "half frequency" (curve 3 in Fig. 2)

$$\omega_p = \frac{1}{2} \omega = \sqrt{kg + k^3 (\sigma/\rho)} \, . \quad (2)$$

For water we would have[12] $\sigma/\rho = 73 \text{ cm}^3/\text{s}^2$, and, as shown in Fig. 2, the pairs of values (ω, k) which are detected experimentally (where $k = 2\pi/\lambda$, and λ is the observed period of the lattice agree well with this "parametric" dispersion relation. However, the linear approximation in (1) does not explain the type of lattice or the steady-state level of the waves, which are determined by nonlinear effects, as in crystals.

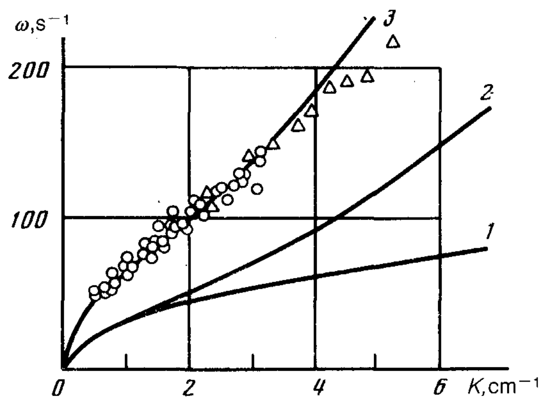

FIG. 2. Comparison of the dispersion relations for (1) gravity waves, (2) gravity-capillary waves and (3) parametric waves with the measured parameters of (triangles) hexagonal cells and (circles) square cells of wave structures.

For a nonlinear analysis we describe the surface by $z = z_0 = H_0(t) + H_1(x,y,z,t)$ and use the equations

$$\left.\begin{aligned} &\mathbf{v} = \nabla\Psi,\ \operatorname{div}\mathbf{v} = 0,\ [\dot{\Psi} + \frac{1}{2}v^2 + gz + (p/\rho)]_{z_0} = F_0(t) \\ &p\big|_{z_0} = p_{\text{atm}} - \sigma\operatorname{div}(\mathbf{G}/\sqrt{1+Q}),\ v_z\big|_{z_0} = v_0 + \dot{H}_1 + (\mathbf{G}\mathbf{v})_{z_0} \end{aligned}\right\}, \tag{3}$$

where we are using $\mathbf{G} = \widehat{\nabla} H_1$, and $Q = G^2$.

3. In an attempt to solve this system of equations we choose the potential

$$\Psi(x, y, z, t) = zv_0(t) + \psi,\quad \psi(x, y, z, t) = f(t)R(x, y)\exp(kz). \tag{4}$$

If we also introduce the dimensionless coordinates $\xi = kx$, $\eta = ky$, $\vec{\rho} = \xi\mathbf{e}_x + \eta\mathbf{e}_y$, using the notation $\widehat{\nabla}_\perp = \partial/\partial\vec{\rho}$, and introduce the two functions $h = kH_1$ $S = k^2\psi|_{z_0}$, we find that the Bernoulli integral in (3) leads to the equation

$$\dot{S} + k(g + \delta g)h - \frac{\sigma}{\rho}k^3\operatorname{div}_\perp(\mathbf{G}/\sqrt{1+Q}) = \frac{1+Q}{2}S^2 - \frac{1}{2}(\nabla_\perp S)^2 + F_1(t), \tag{5}$$

where $F_1(t)$, like $F_0(t)$ in (3), is arbitrary but is to be determined. Boundary condition (3), on the other hand, gives us the relation $S = (h + \mathbf{G}\widehat{\nabla}_\perp S)/(1+Q)$. If the amplitude is small ($h \ll 1$), we can express S in terms of h by the following expansion, which holds within cubic terms:

$$S = S_1 + S_2 + S_3 + \ldots,\quad S_1 = \dot{h},\quad S_2 = \frac{1}{2}\dot{Q},\quad S_3 = \frac{1}{2}\mathbf{G}\nabla_\perp\dot{Q} - Q\dot{h}. \tag{6}$$

Subsituting this expansion into (5), we finally find an equation for h:

$$\left.\begin{aligned} &\ddot{h} + k(g + \delta g)h - \frac{\sigma}{\rho}k^3\Delta_\perp h = A_2 + A_3 + F_1(t),\quad A_2 = \frac{1}{2}(\dot{h}^2 - \ddot{Q} - \dot{\mathbf{G}}^2) \\ &A_3 = Q\ddot{h} + \frac{3}{2}\dot{Q}\dot{h} - \dot{\mathbf{G}}\nabla_\perp\dot{Q} - \frac{1}{2}\mathbf{G}\nabla_\perp\ddot{Q} - (\sigma k^3/2\rho)\operatorname{div}_\perp(Q\mathbf{G}) \end{aligned}\right\} \tag{7}$$

This equation determines the type of lattice.

4. We seek a solution of (7) as a sum of stationary waves:

$$h = T(t)R(\xi, \eta),\quad R = R_N = \sum_i^N c_i,\quad c_i = \cos(\mathbf{n}_i\vec{\rho}), \tag{8}$$

where $\mathbf{n}_i$ are unit vectors along the various directions. For example, from the set of unit vectors

$$\mathbf{n}_0 = \mathbf{e}_x,\quad \mathbf{n}_1 = \mathbf{e}_y,\quad \mathbf{n}_{2,3} = -\frac{1}{2}(\pm\sqrt{3}\,\mathbf{e}_x + \mathbf{e}_y),\quad \mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3 = 0 \tag{9}$$

we can construct either a one-dimensional set of ridges, $R_1 = c_0$, or a "parquet"; the parquet may be one of either squares, $R_2 = c_0 + c_1$, or hexagons, $R_3 = c_1 + c_2 + c_3$. Substituting $h = TR_N$ into (7), we find the expressions

$$A_2 = \frac{1}{2}\dot{T}^2R^2 - (T\ddot{T} + \frac{3}{2}\dot{T}^2)(\nabla_\perp R)^2, \quad A_3 = T[T\ddot{T} + 3\dot{T}^2 + T^2(\sigma k^3/2\rho)]D, \qquad (10)$$

where $D = -\operatorname{div}_\perp[(\nabla_\perp R)^2\nabla_\perp R]$. Since the left side of (7) is proportional to R, however, we should retain only the cosines on the right side which appear in R, discarding all of the other, "nonresonant," cosines. As a result, we find the following "replacement rules" for the operators in (10):

$$D_N \to \frac{3}{4}R_N\delta_N, \quad R_N^2 \to R_N\delta_{N,3}, \quad (\nabla_\perp R_N)^2 \to \frac{1}{2}R_N\delta_{N,3}, \qquad (11)$$

where $\delta_N = 1 + (1/3)(N^2 - N)$, and $\delta_{N,3} = 1$ at $N = 3$ or $\delta_{N,3} = 0$ at $N \neq 3$.

At this point, it is convenient to introduce the dimensionless time $\tau = \omega t/2$; the parameters

$$\alpha = \sigma k^3/\rho\omega^2, \quad a = 4(kg\omega^{-2} + \alpha), \quad q = 2kH, \quad \epsilon = 2q/a; \qquad (12)$$

and the function $\gamma(\tau) = 1 - \epsilon\cos 2\tau$. From (7) we then find the equations

$$M = M_3\delta_N \text{ for } N = 1,2; \; M = M_2 + 3M_3 \text{ for } N = 3, \qquad (13)$$

where M is the "standard" Mathieu operator $M = \ddot{T}(\tau) + a\gamma(\tau)T$. Here

$$M_2 = -\frac{1}{2}T\ddot{T} - \frac{1}{4}T'^2, \quad M_3 = \frac{3}{4}T(T\ddot{T} + 3\dot{T}^2 + 2\alpha T^2). \qquad (14)$$

It can be seen from these equations that at small values $T \ll 1$ only a hexagonal lattice ($N = 3$) is possible; at moderate values ($T \lesssim 1$), we can have only a square lattice ($N = 2$); and at $T \gtrsim 1$, we can have only a one-dimensional lattice ($N = 1$). Experiments confirm this sequence of "phase transitions," i.e., changes in lattice type (Fig. 3), although we did not study these changes in detail.

5. Let us examine a hexagonal lattice in more detail. For simplicity, we assume $T \ll 1$, and $M_3 \ll M_2$. From (13) we then find the equation

$$M = M_2, \quad (1 + \frac{1}{2}T)T'' + \frac{1}{4}T'^2 + a\gamma T = 0, \qquad (15)$$

which can be solved only by numerical methods. In the limit $\epsilon \to 0$, it has a solution in the form of the elliptic integral $E(\varphi, l)$ with $|T| \leqslant T_* = T_{\max} = 2$:

$$\tau\sqrt{a} = 2E(\varphi, k)\sqrt{2/(2-k^2)}, \quad k = \sqrt{2T_*/(2 + T_*)}, \quad \varphi = \arcsin\sqrt{(T_* - T)/2T_*}. \qquad (16)$$

In this case, the condition for "half-synchronization" with the vibrations of the bottom determines the parameter

$$a = 8\pi^{-2}E^2(k)/(2 - k^2) \text{ for } 8\pi^{-2} = a_{min} < a < a_{max} = 1 \qquad (17)$$

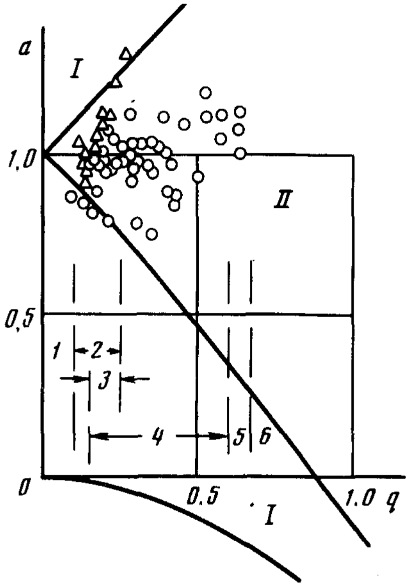

FIG. 3. Representation of the process by which the lattices form in the stability region (I) and the instability region (II) of the solutions of the Mathieu equation. 1—No lattices; 2—hexagonal cells; 3—region of "phase transitions" in the lattices; 4—square cells; 5—ridge cells; 6—chaos.

as a function of the maximum amplitude $T_{\max}$, which is therefore independent of ϵ at $\epsilon \ll 1$. Actually, however, it is ultimately determined by specifically the value of ϵ. For the square and one-dimensional lattices, we can also find a "half-synchronization" condition similar to (17) in the limit $\epsilon \to 0$, but we will not pursue that point here.

Experiments confirm these theoretical conclusions. In practice, the empirical relation $H_1^{\max} = \lambda/3$ holds approximately; this relation does not include the vibration amplitude of the bottom, H (Ref. 1).

To derive the "law" $H_1^{\max} \gtrsim \lambda/3$ from the theory, we should find an exact solution of (13), but the "law" $H_1^{\max} \gtrsim \lambda/3$ also indicates that the dimensionless amplitude $h^{\max} = kH_1^{\max} = 2$ is never small. Strictly speaking, therefore, we cannot expand the solution in a series in this amplitude, although this is precisely what we did above. Consequently Eqs. (13) should be regarded as being of only "qualitative" validity. A more rigorous expansion in powers of h was carried out in Ref. 10.

The fact that there are no small ripple amplitudes in a "vibrating gravitational field" fundamentally distinguishes our case from that of the steady-state structures,[5–9] where small amplitudes are possible if the fields are only slightly above the critical level.

The finite size of the depth of the layer, Z, can be incorporated in our equations through the substitution $g \to g \tanh kZ$, which makes the experimental results dependent on the depth of the layer (Fig. 3).

Since the decay of the wave structure described above may give rise to gravity waves, the evolution of a seismic disturbance at sea should be regarded as a possible mechanism responsible for the formation of tsunamis.

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