MHD stability of nonlocal quasiflute modes in closed magnetic confinement systems

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An analytic minimization of a potential-energy functional of MHD perturbations in a finite-pressure plasma is carried out. In closed systems, nonlocal quasiflute modes can develop. They turn out to be more dangerous than localized perturbations. As an example, a necessary and sufficient condition for MHD stability is derived for confinement systems of the DRAKON type. This condition is sufficient for arbitrary current-free paraxial systems with circular magnetic surfaces.

1. One of the most common methods for analyzing the stability of a plasma is based on an energy principle,[1] according to which a necessary and sufficient condition for the MHD stability of a plasma with a pressure p and an adiabatic index γ in a magnetic field $\mathbf{B}$ with respect to arbitrary small displacements $\vec{\xi}$ is that the following energy functional be positive definite[2]:

$$W = \frac{1}{2} \int dV \, \{\mathbf{T} + (\vec{\xi}\mathbf{n})\Lambda\mathbf{n})^2 + (\vec{\xi}\mathbf{n})^2 \, [\mathbf{B}\vec{\nabla}\Lambda - 2\Lambda\mathbf{n}\,\mathrm{rot}\,[\mathbf{nB}] - \Lambda^2 - K] + \gamma p \, \mathrm{div}^2 \vec{\xi}\,\} . \qquad (1)$$

Here $\mathbf{T} = \mathbf{Q} + (\vec{\xi}\mathbf{n})$ [curl $\mathbf{Bn}$]; $\mathbf{Q}$ is the perturbed magnetic field (in a plasma, $\mathbf{Q} = \mathrm{curl}[\vec{\xi}\mathbf{B}]$); $K = 2[\mathrm{curl}\,\mathbf{Bn}]\ (\mathbf{B}\nabla)\mathbf{n}$, $\mathbf{n} = -\nabla p/|\nabla p|$ is the unit vector normal to a magnetic surface; and Λ is an arbitrary single-valued function [which is introduced for convenience in the case of closed systems with magnetic surfaces; the sum of all the terms with Λ in (1) is zero]. The usual procedure for determining the most

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dangerous perturbations is to minimize W with respect to $\vec{\xi}$, but in general (without any further assumptions) this direct minimization leads to extremely complex equations and fails to provide a universal, necessary and sufficient condition for MHD stability.

Let us examine the stability of a low-pressure plasma ($\beta \sim p/B^2 \ll 1$) in a closed, current-free system. In this case, the most dangerous perturbations are incompressible perturbations which are stretched out along the magnetic lines of force (quasiflute perturbations). Any real perturbation must of course be a single-valued function in space (and, correspondingly, a periodic function of the toroidal coordinates), but this requirement generally does not mesh well with the requirement that the perturbation be of a flute nature. The only situation in which there is clearly no contradiction is that in which the lines of force are closed, and this circumstance is usually pursued to the conclusion that the most dangerous perturbations are localized near rational magnetic surfaces. This assumption (and its consequences) imposes rather severe restrictions on the nature of the transverse profiles of these perturbations (transverse with respect to the magnetic field), and it generally yields only a necessary condition for MHD stability.[3,4]

2. It is our view that the class of perturbations which can be analyzed can be increased significantly. The distance (L) over which a perturbation is closed in accordance with the requirement of periodicity may be significantly greater than the scale length l of the changes in the magnetic field; this is the case, for example, in the DRAKON system,[5] the helical torus, and similar systems. It is not difficult to see that for values of β which are not too small, $\beta > (l/L)^2$, the perturbation of the magnetic field imposed by the requirement of periodicity may be comparatively small [in comparison with the term containing K in (1)], even for nonlocal perturbations. At this stage of the minimization, we thus do not need to introduce any further assumptions regarding the nature of the transverse profile of $\vec{\xi}$, so that we can hope to derive a necessary and sufficient condition for MHD stability. The derivation of such a condition is the subject of this letter.

3. Let us describe a formalism for minimizing W. We use the paraxial approximation (expanding all quantities in power series in $\lambda \sim a/l \ll 1: X \approx X_0 + \lambda X_1 + \lambda^2 X_2 + \ldots$, where a is a transverse scale dimension of the plasma). When there are magnetic surfaces (and we are considering only such systems), the longitudinal component of $\vec{\xi}$ appears only in the last term in W (Ref. 6), which reaches a minimum in the case of incompressible perturbations. The longitudinal component of $\mathbf{T}$ is $\mathbf{TB}/B \sim BX/a$ (where $X \sim |\vec{\xi}|$); it is the leading term in the integrand. Consequently, in each step of the minimization it is minimized independently by virtue of higher-order displacements. As a result, the condition $\mathbf{TB}/B \approx 0$ should hold with an accuracy to within terms $\sim XB\lambda^3/a$. This accuracy is achieved by choosing $\vec{\xi}$ in the form

$$\vec{\xi} = \frac{1}{B^2}\,\mathrm{curl}(BF\mathbf{B}) - \frac{\mathbf{B}}{B^3}\,F(\mathbf{B}\,\mathrm{curl}\,\mathbf{B}) - 2F\,\frac{(\mathbf{n}\vec{\nabla} p)}{B^3}\,[\mathbf{B}\mathbf{n}], \qquad (2)$$

where $F \sim Xa$ is an arbitrary function (we are assuming $p/B^2 \sim \lambda^2$).

Substitution (2) makes it possible to minimize W with respect to the single function F and to thereby find, through numerical calculations, an accurate stability condition for an arbitary given system. An analytic derivation of the condition in this general form, in contrast, is difficult, because the corresponding Euler extremal is described by a rather complex fourth-order partial differential equation. We will accordingly continue the procedure of a sequential minimization of W in the paraxial approximation. This procedure requires that we cause the leading-order terms $(XB\lambda/a)$ in the transverse components of **T** to vanish, but it does not lead to trivial perturbations with $(\vec{\xi}_0\mathbf{n}) = B^{-2}[\mathbf{Bn}]\vec{\nabla}(F_0B) \approx 0$, which are of no interest (nontrivial perturbations might be, for example, perturbations which are localized near rational magnetic surfaces). Nevertheless, with $l/L \lesssim \lambda$ and $f = B^{-2}\mathbf{B}\vec{\nabla}(FB) \sim \lambda X(l/L ... + \lambda ... + ...)$, despite the fact that a perturbation of the transverse magnetic field is formally nonzero in the leading order, the corresponding term in the integrand can be quite small (of the next higher order in λ or even smaller) because of the appearance of a new, λ-independent parameter l/L. Consequently, the logic of the expansion in powers of λ is retained. In the plasma region, the functional in (1) takes the form

$$W_{pl} \approx \frac{1}{2} \int B^2 dV \{(B^{-1}[\mathbf{Bn}]\vec{\nabla}f + (\vec{\xi}_0\mathbf{n})\Lambda B^{-1})^2 + (\mathbf{n}\vec{\nabla}f - (\vec{\xi}_0\mathbf{n})B^{-2}(\mathbf{B}\,\mathrm{curl}\,\mathbf{B}))^2 + (\vec{\xi}_0\mathbf{n})^2 B^{-2}[|\vec{\nabla}p|^2 \mathbf{B}\vec{\nabla}(\Lambda|\vec{\nabla}p|^{-2}) - \Lambda^2 - K]\}. \tag{3}$$

The quantity $\Lambda \sim B\lambda^2/a$ is chosen to cancel the terms $\sim B^2\lambda^3/a^2$ in the expression in square brackets in (3) (such a value of Λ does not exist). As a result, all the terms in the integrand in (3) are of the same order of magnitude ($\sim B^2\lambda^4/a^2$), and it is in this order that the condition should be derived. The quantity $f = B^{-2}\mathbf{B}\vec{\nabla}[B(F_0 + \lambda F_1)]$ is varied in (3) in a manner independent of $(\vec{\xi}_0\mathbf{n})$, but under the solvability condition $\int fB^2 dV \approx 0$ (an integration between magnetic surfaces).

To determine the value of f at the boundary magnetic surfaces (from the transversality condition), we should in general add to (3) an integral over the vacuum region, $W_{\mathrm{vac}} = \int \mathbf{Q}^2 dV/2$, joining the field at the plasma boundary.

The last (and most laborious) step of the minimization, involving the quasiflute part of the displacement $(\vec{\xi}_0\mathbf{n})$, is carried out without any preliminary restrictions on the radial profile of $(\vec{\xi}_0\mathbf{n})$. Consequently, this step may lead to a stability condition which is more restrictive than the conditions for local stability which have been derived previously.

4. Since it is difficult to minimize W analytically in invariant form, we will illustrate the process with a result derived for paraxial systems with magnetic surfaces of circular cross section[1)] and a parabolic pressure profile $p = \lambda^2 p_0(1 - \Psi/\Psi_b)$, where $\Psi \approx r^2 B_0 + \lambda ...$, $\Psi_b = \mathrm{const}$, r is the radius of the magnetic surface, and $B_0(s)$ is the magnetic field at the axis of the system [with a curvature $k(s)$ and a twisting $\kappa(s)$]:

$$\oint \frac{ds}{B_0^2} \left\{ -\frac{3}{4}\left(\frac{B_0'}{B_0}\right)^2 - k^2 + k\frac{B_0'}{2B_0}(I_u \cos\alpha + I_g \sin\alpha) \right\} - \beta_1 B_0 \sqrt{\overline{G_u^2 + G_g^2}} > 0. \tag{4}$$

Here $G_u = \oint ds(I_u^2 - I_g^2)/B_0^3$; $G_g = \oint ds(2I_uI_g)/B_0^3$; $\beta_1 = 2a^2\lambda^{-2}B_0^{-1}(\partial p/\partial\Psi) \sim 1$; $(I_u/B_0^{3/2})' = k\cos\alpha/B_0^{3/2}$; $(I_g/B_0^{3/2})' = k\sin\alpha/B_0^{3/2}$; and $\alpha' = \kappa$ (the prime means the derivative with respect to s, i.e., along the axis of the system). Condition (4) is a necessary and sufficient condition for stability of the plasma in the DRAKON confinement system[5] with a long (but not infinite) straight section. It may also be interpreted as a sufficient condition for any closed current-free system with a three-dimensional axis. The last term in the integrand in (4) clearly distinguishes condition (4) from the Mercier and ballooning-mode condition.

5. The perturbations which we have analyzed here, and which correspond to condition (4), are quasiflute displacements which are modulated by a weaker structure of a ballooning type. In terms of azimuthal profile, they are a wide packet of even or odd harmonics, respectively, coupled by toroidal effects (the stability condition for an individual mode is the same as the Mercier condition). The radial profile can have rather strong oscillations; the region of localization along the radial direction is determined by the shear (if the shear is weak, this region becomes the entire plasma column). To analyze the structure of the resulting perturbations in more detail would be to go beyond the scope of the present paper.

[1)] Such systems may have an "average min B," as was shown in Ref. 7.

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