Generation of ordered structures with a symmetry axis from a Hamiltonian dynamics

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A mapping which generates ordered structures with a symmetry axis of arbitrary order is constructed. This generator results from the motion of a particle in a potential with two competing symmetries. As example, planes are tiled with fivefold and sevenfold symmetry axes.

The problem of the mechanism for the stochastic dynamics of particles in a static magnetic field and in the field of a wave packet which is propagating across the magnetic field leads to new and, at first glance, unexpected consequences. These consequences not only bear directly on the theory of Fermi acceleration but also present an analogy with the symmetry properties of hydrodynamic and plasma structures and the structures of quasicrystals. The results reported below establish this relationship.

The Hamiltonian describing the motion of a particle is

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$$H = \frac{m}{2} (\dot{x}^2 + \omega_H^2 x^2) - e \sum_{n=-\infty}^{+\infty} \frac{1}{k} E_k \cos(kx - n\Delta\omega t), \tag{1}$$

where ω_H is the Larmor frequency, and $\Delta\omega$ is the interval between the frequencies of the wave packet. In the case with $E_k = \text{const} = E_0$ and $k = k_0$ (a uniform packet), the equations of motion generated by Hamiltonian (1) lead exactly to a mapping with a twisting through an angle α (Ref. 1):

$$\hat{M}_\alpha: \begin{cases} \bar{u} = [u + (K/\alpha) \sin v] \cos \alpha + v \sin \alpha \\ \bar{v} = -[u + (K/\alpha) \sin v] \sin \alpha + v \cos \alpha , \end{cases} \tag{2}$$

where $K = ek_0E_0T^2/m$, $\alpha = \omega_H T$, $u = k_0\dot{x}/\omega_H$, and $v = -k_0x$; the step of the mapping is $T = 2\pi/\Delta\omega$. We showed in Ref. 1 that under the resonance condition

$$\alpha = 2\pi p/q, \tag{3}$$

where p and q are integers ($p<q$), the (u,v) phase plane becomes covered with a stochastic web at arbitrarily small values of K. This web forms at the locations of the destroyed separatrices and has the approximate symmetry of rotation through an angle of $2\pi/q$.

There is a "window" near the center of the separatrix grid which is bounded by an invariant curve. In this window there is a complex substructure with a symmetry different from that of the web outside the window. The size of the window increases as the parameter K decreases.

At this point we introduce the dimensionless time $\tau = t/T$, which should appear in the canonical equations of motion.

Equation (2) and the corresponding Hamiltonian

$$H_\alpha = \frac{1}{2} (u^2 + v^2) - (K/\alpha^2) \cos v \sum_{n=-\infty}^{+\infty} \delta(\tau - n) \tag{4}$$

have a remarkable property: The first term in (4) describes paths in the (u,v) plane with a symmetry axis of arbitrary order. The second term in (4) describes paths with translational symmetry with respect to the displacement $v \to v + 2\pi s$ (s is an integer). The combination of these two terms leads to an interaction of a rotational symmetry with a translational symmetry; this interaction is strongest at the resonance in[1)] (3). The mapping $\hat{M}_q$ in this case becomes a generator of a "tiler mapping," i.e., a coverage of the plane with ordered structures, with a q-fold symmetry axis. This symmetry is manifested in the structure of the stochastic web and in the arrangement of islands of stability within the cells of the separatrix. We should add that the symmetry becomes more regular as K decreases.

Examples of symmetric coverages of this sort with $q = 3$, 4, and 5 are given in Ref. 1. One can see a correspondence between the structure in the (u,v) phase plane, which is generated by M_5 and Penrose coverages.[2] As a result, there is a direct analogy between the paths of a particle with the Hamiltonian $\hat{H}_5$ and the structure of the quasicrystals which were discovered in the experiments of Ref. 3.

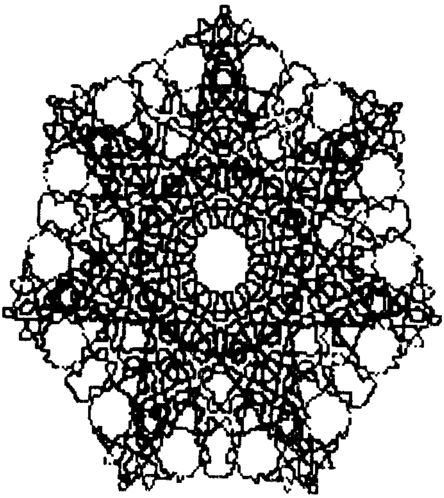

FIG. 1. Stochastic web formed by the mapping $\hat{M}_7$ with $K/\alpha = 0.5$. The size of the square enclosing the "star" is $64\pi \times 64\pi$.

Here is a nontrivial example of the generation of a coverage with a sevenfold symmetry axis. Figure 1 shows the stochastic web formed by the Hamiltonian H_7 or by the equivalent mapping $\hat{M}_7$. This web consists of all points on one path which lies within a narrow region of stochastic dynamics. This web can be "deciphered" in a definite manner by passing lines through its nodes in such a way that an ordered parquet is somehow produced. In particular, one example of a parquet of this type would consist of regular heptagons and three additional elements.

The appearance of a fivefold symmetry in quasicrystals was explained on the basis of Landau's arguments regarding as expansion of the energy in powers of the order parameter.[4] Some slightly different ideas were also used in Refs. 5 and 6. So far, however, we do not have a model of a real physical system which generates structures with a fivefold symmetry. In the case at hand, we can work from first principles (i.e., from the Hamiltonian H_α or the equations of motion for $\hat{M}_\alpha$) to construct a Hamiltonian whose paths cover the phase plane with a parquet with some arbitrary prespecified rotational symmetry. For this purpose, it is sufficient to single out the resonant terms in (4) with $\alpha = 2\pi/q$. Straightforward calculations lead to (in a coordinate system rotating at a frequency ω_H)

$$\tilde{H}_q = \bar{H}_q + H_{int}\,; \quad \bar{H}_q = -\frac{K}{\alpha^2 q}\sum_{k=1}^{q}\cos\left(v\cos\frac{2\pi k}{q} + u\sin\frac{2\pi k}{q}\right); \tag{5}$$

$$H_{int} = -\frac{k}{\alpha^2}\sum_{n\neq kq} J_n\left(\sqrt{u^2+v^2}\right)\cos\left\{\frac{n}{q}\arctan\frac{v}{u} + 2\pi\left(\frac{n}{q}-k\right)\tau\right\},$$

where J_n is the Bessel function. The case $\bar{H}_q$ with $q = 3$ and 6 determines Bénard cells in heat convection.[7] The case $q = 2$ corresponds to convection spindles. The case $q = 4$ was studied in Ref. 1. The expression found from $\bar{H}_q$ with $q = 5$ was given in Ref. 6 as an order parameter. Expression (5) thus not only contains familiar structures of a continuous medium but also contains some possible structures with arbitrary q. Furthermore, expression (5) contains the form of the perturbation potential of these structures,[2)] H_{int}.

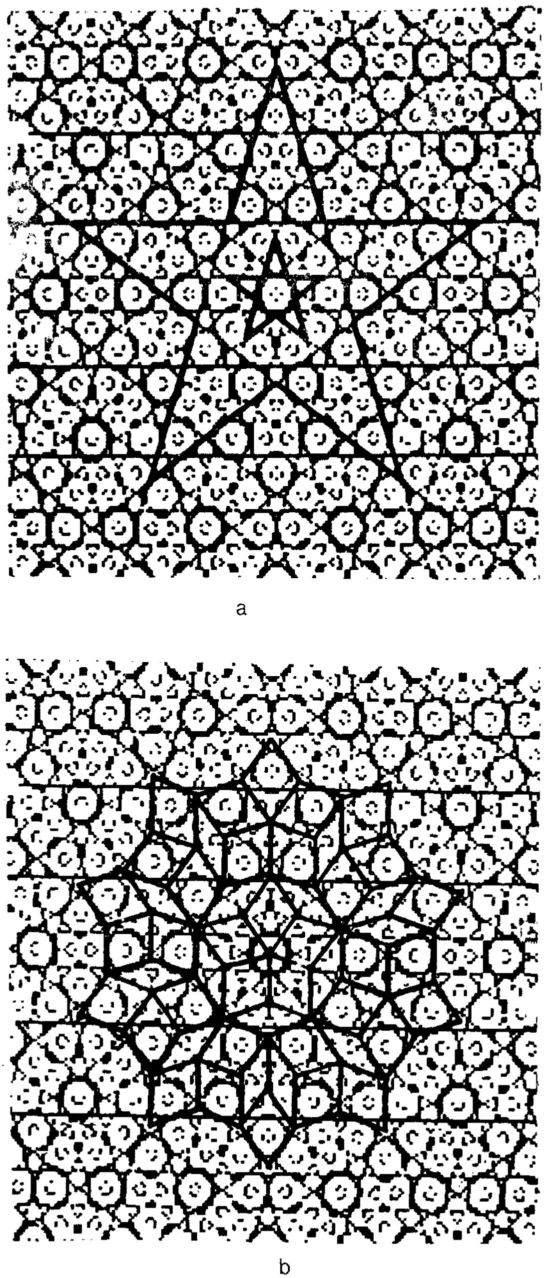

FIG. 2. Energy relief map of an $\overline{H}_5$ particle near separatrices. The size of the square is $64\pi \times 64\pi$. a—Relief map; b—the same on which a Penrose coverage has been superimposed (the algorithm for determining the points which are to be connected is obvious from this figure).

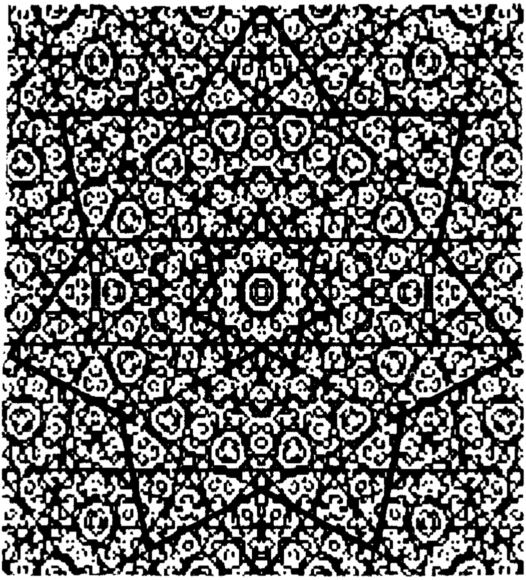

FIG. 3. Energy relief map of an $\overline{H}_7$ particle near separatrices. The size of the square is $80\pi\times 80\pi$.

Using expression (2) with $\alpha = 2\pi/q$, i.e., for a tiler mapping, we find the following:

1) There exists a coverage with q-fold symmetry of the (u,v) plane.

2) The structure of the coverage is given approximately by a stochastic web (i.e., by a separatrix grid) of the Hamiltonian $\overline{H}_q$.

3) The perturbation $H_{\rm int}$ dresses this grid with a coverage by a stochastic layer $\sim\exp(-\text{const}/K)$.

4) The generator in (2) which covers a plane can be generalized to three dimensions by introducing corresponding Euler angles.

5) The structure of the coverage can be found approximately by a q-fold rotation of an infinite system of parallel lines of nonzero (!) thickness.

Figures 2 and 3 are energy relief maps for a particle. The hatching corresponds to the energy (E) of a particles whose path passes near a separatrix. These relief maps indicate a region in which the particle "percolates" along the channels of the stochastic web, and they correspond to fivefold and sevenfold symmetries. A Penrose coverage is superimposed on the relief of the percolation regions in Fig. 2b.

It can be seen from Figs. 2 and 3 that the symmetry of the coverage has a fractal property in the direction of an increase in the coverage elements. Similar figures of increasing size are generated (see the emphasized pentagons and heptagons in Figs. 2a and 3).

In summary, the tiler mapping $\hat{M}_q$ realizes a relationship between a coverage of a plane with a symmetry axis of arbitrary order q and an equivalent dynamic system (4) with $\alpha = 2\pi/q$, in which special paths of a particle within the channels of the stochastic web perform the required coverage.

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[1)]For simplicity, at this point we set $p = 1$, and we denote H_α and $\hat{M}_\alpha$ at $\alpha = 2\pi/q$ by H_q and $\hat{M}_q$, respectively.

[2)]The complete Hamiltonian $\tilde{H}_q$ has an approximate $2\pi/q$-fold rotational symmetry. On the other hand, its average part $\overline{H}_q$ consists of two identical, superimposed coverages with q-fold symmetry. The resultant coverage for $\overline{H}_q$ is therefore symmetric with respect to rotation through π/q for odd q.

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