Symmetry of instanton solutions in gauge theories

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A connection is established between solutions of the instanton type in gauge theories and the transformation properties with respect to transformations from the homogeneous Lorentz group.

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Solutions of the instanton type[1-3] in gauge theories can be due to the fact that the transformation properties of the field quantities with respect to transformations from the homogeneous Lorentz group $O(1,3)$ [in Euclidean space $O(\phi)$] are not uniquely defined, and that different possibilities correspond to different solutions that are "symmetrical" with respect to the Lorentz group and are subsequently attributed to spontaneous breaking of the vacuum state. Indeed, if a gauge theory is Lorentz-invariant for some definite choice of the transformation properties, and the gauge group contains the Lorentz group $O(\phi)$ as a subgroup, then, by carrying out simultaneously with the transition to the Lorentz system also a rotation of the gauge components with respect to $O(\phi)$, we alter by the same token the transformation properties of the field quantities without changing either the action or the equations of motion, since they are invariant to transformations from the gauge group.

The situation recalls in obvious fashion the physics of second-order phase transitions, namely, several symmetries, or more accurately transformations, are possible in the theory, and the dynamics of the process selects one of them under the appropriate conditions.

It is convenient to illustrate the foregoing by using as an example the Yang-Mills theory with a triplet of vector particles. In the usual variants the theory contains three vector particles that differ from one another by the calibration indices. The three generators L_i and SU(2) form a three-dimensional representation of the rotation groups of the four-dimensional space $O(\phi) = \mathrm{SU}(2) \times \mathrm{SU}(2)$, and consequently, if we apply to the potentials A_i (which transform in accordance with the representation $(1/2, 1/2)$), simultaneously with the Lorentz transformation, an additional rotation in the gauge indices, then the transformation properties of the theory will be determined by the representation $(\frac{1}{2}\,\frac{1}{2}) \otimes (10) = (\frac{3}{2}\,\frac{1}{2}) \oplus (\frac{1}{2}\,\frac{1}{2})$ of the four-dimensional rotation group. A gauge transformation can be used to eliminate the three field components, after which we are left in the theory with particles having spins 2, 1, 0 in place of the triplet of particles with equal spins in the initial variant.

If we seek a solution of the Yang-Mills equations that is invariant (having the same form in different reference frames) relative to the group Lorentz, then the only possibility in the first variant is the following choice of the potentials: $A_i = (O\phi^\alpha / Ox_i) L_\alpha$ which leads automatically to zero values for the field tensor. In the second variant, besides the invariant vectors x_i and $\partial/\partial x_i$, there exists an antisymmetrical tensor of second rank $F_{ij} (F_{\alpha\beta} = \epsilon_{\alpha\beta\gamma} L_\gamma,\ F_{\alpha\gamma} = L_\alpha)$, and the most general form of the invariant potential is

$$A_i = f_1 (F_{ik} x k) + f_2 F_{ik} \frac{\partial f_3}{\partial x_k} + \frac{\partial f_4}{\partial x_i} \left(x_j F_{jn} \frac{\partial f_5}{\partial x_n} \right),$$

where f_α are certain scalar functions determined by the system of equations of the gauge theory. By choosing a particular form of the vector potential

$$A_i = F_{ik} \frac{\partial \phi}{\partial x_k}$$

we obtain as a consequence of the duality conditions $F_{ij} = -\tilde{F}_{ij}$ an equation for ϕ in the form

$$\Box \phi - \left(\frac{\partial \phi}{\partial x_i} \right)^2 = 0 \qquad \text{or} \qquad e^{\phi} \Box e^{-\phi} = 0.$$

Thus, in particular,

$$e^{-\phi} = \sum_a \approx \frac{e_a^2}{R_a^2} + \frac{1}{\lambda}, \qquad R_a^2 = (x - x_a)^2, \qquad 1/\lambda = \text{const}. \tag{1}$$

The previously obtained[2] solutions of the instanton type follow directly from (1) if a central-symmetry solution is chosen: $e^{-\phi} = (1/R) + (1/\lambda)$. A_i is given by the expressions $A_i = -[2\lambda/\tau(\tau+\lambda)](F_{ij} x_j)$ which are obtained from the solution of[2] by an inversion transformation $[x_i \to (x_i/x^2)]$ (the duality condition reverses sign under inversion). The topological charge is numerically equal to the number a of sources in the expression (1) and does not depend on their charges $(e_a^2 > 0)$.

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