Comparison of topological aspects of gauge fixing in a string theory and a Yang-Mills theory

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The topological obstacles to gauge fixing are fundamentally different in a string theory and in a Yang-Mills theory. In a string theory, the principal stratification, determined by the action of the group of diffeomorphisms, is reducible to a compact or discrete group, while in a non-Abelian theory the corresponding reduction is not possible.

Let us compare the topological aspects of the specification of the gauge in a Polyakov string theory[1] and in a non-Abelian field theory. In either case, a global gauge is impossible, but in a string theory there globally exists an incomplete gauge with a residual symmetry which reduces to the action of a compact or discrete group—or a combination of the two. We will show that in a non-Abelian field theory the corresponding gauge is topologically forbidden. This result reinforces Singer's result[2] on Gribov indeterminacies[3].

We know that the action

$$\frac{1}{4\pi\alpha'}\int_S d^2\xi\sqrt{g}g^{ab}\partial_a x^\mu\partial_b x_\mu \qquad (1)$$

[ξ are the coordinates of the world surface of the S-string, g is the metric on it, and $x^\mu(\xi)$ is the insertion of S in a d-dimensional space-time] is invariant under the group $D(S)$ of orientation-preserving diffeomorphisms and under the Weyl group $g\to\lambda g, \lambda(\xi)>0$, which we denote by $W(S)$. As in a Yang-Mills theory, the action of the gauge-transformation group G (in terms of the modulus of the center) determines the structure of the principal stratification in the space of irreducible gauge potentials $\widetilde{P}$; the action D determines a structure of this sort in the space $\widetilde{M}$ of metrics with trivial isometries. This stratification again has no global cross section, since otherwise the relation

$$\pi_i(\widetilde{M})=\pi_i(D)\times\pi_i(\widetilde{M}/D) \qquad (2)$$

would hold, but it could not hold since $\widetilde{M}$, like $\widetilde{P}$, has null homotopic groups, while D is homotopically nontrivial, like G. The analogy with a Yang-Mills theory was drawn in Ref. 4. In the present letter we examine the differences that arise because the stratification $\widetilde{M}/D$ is amenable to a far-reaching reduction. This fact follows from the results of Ref. 5, derived in the theory of Teichmueller spaces,[6] which is presently being used widely to study multiloop amplitudes.[7–12] In the case of an S^2 sphere, the group $D(S^2)$ as a topological space can be represented in the form

$$D(S^2)=H\times D(S^2;x_1,x_2,x_3), \qquad (3)$$

where $H=SL(2,\mathbb{C})/\mathbf{Z}_2$ is the group of holomorphic automorphisms, while $D(S^2;x_1,x_2,x_3)$ is the subgroup which leaves three different points of the sphere in place. This subgroup acts freely both on the space of metrics M and the space of complex structures $C=M/W$, both of which can be contracted. By reducing the problem to a Beltrami equation, one can easily show that its action on $C(S^2)$ is transitive.[5] For the stratification $M/D(S^2;x_1,x_2,x_3)$ this result means that the surface Σ of metrics, which are conformal with the standard, serves as a global cross section of it (its continuity follows from the continuous dependence of the solution of the Beltrami equation on the coefficients). For $\widetilde{M}/D$, the intersection $\Sigma\cap\widetilde{M}$ determines the reduction to H. Since H contains SO(3) as a maximal compact subgroup, a further reduction to SO(3) is also possible. In the case of the torus T^2, there is an analogous representation for the component of unity:

$$D_0(T^2)=H_0\times D_0(T^2,x_0). \tag{4}$$

The subgroup $D_0(T^2,x_0)$ again acts freely, and it is not a difficult matter to construct the explicit cross section for its action in M and C. Specifically, if we treat T^2 as a factor space of the complex plane $\mathbb{C}$ on an integer lattice, for an arbitrary metric there exists a unique transformation from $D_0(T^2,x_0)$ which puts it in the form

$$\lambda(\xi)\,|d\xi_1+\tau d\xi_2|^2,\quad \tau\in\mathbb{C}^+ \tag{5}$$

(λ is a doubly periodic function on $\mathbb{C}$). In other words, the space of orbits for the action $D_0(T^2,x_0)$ on $C(T^2)$ can be identified with the upper $\mathbb{C}^+$ half-plane. It follows that this stratification is trivial, that $D_0(T^2,x_0)$ can be contracted, and that $D_0(T^2)$ is homotopically equivalent to the group $H_0=SO(2)\times SO(2)$ (Ref. 5). When this identification is made, the action of the group D/D_0 on C/D_0 becomes the standard action of the Klein modular group $SL(2,\mathbf{Z})$ on $\mathbb{C}^+$. For the stratification $\widetilde{M}/D$, a surface of metrics of the type in Ref. 5 determines a reduction to a semidirect product of the compact group H_0 and the discrete group $SL(2,\mathbf{Z})$. In this case of a surface of type $p>1$, the analysis of the stratification is similar in many ways to that of a torus. It stems from a realization of S_p as an $\mathbb{C}^+$ factor space in terms of the action of the corresponding Fuchs group. Here, the component of unity D_0 itself acts on M and C freely. The Teichmueller space $T=C/D_0$ is homeomorphic with $\mathbf{R}^{6p-6}$. Hence the stratification $C\to C/D$ is trivial, and $D_0(S_p)$ can be contracted. The action of the Teichmueller modular group D/D_0, which is discrete and which acts in a characteristic discontinuous way,[5,6] is defined at the corresponding cross section.

The group D_0 can thus always be contracted to a compact subgroup (in the case $p>1$, even to unity), making $\widetilde{M}/D$ reducible. We can show that in a non-Abelian field theory the situation is different, and the component of unity of the $\widetilde{G}$ group (the factor group of G with respect to its center) has no homotopic type of a compact Lie group. This result means that an incomplete global gauge which leads to orbits of *finite volume* in the path integral is impossible. We are restricting the analysis to the case, studied in Ref. 2, of the SU(N) gauge group and the single-point compactification of the space-time S^4. The reader is referred to Ref. 13 with regard to the functional class to which the fields must belong. We denote by $G(x_0)$ the subgroup of gauge transformations which are equal to one at the point x_0. This subgroup is normal in G, and we have

$$G/G(x_0)=SU(N). \tag{6}$$

For its homotopic groups we can write[2]

$$\pi_i G(x_0)=\pi_{i+4}SU(N). \tag{7}$$

Let us assume that $\widetilde{G}_0$ has a homotopic type of the compact group H. We then have

$$\pi_i G=\pi_i H \text{ for } i>1. \tag{8}$$

Substituting (7) and (8) into the exact homotopic stratification sequence (6), we find

$$\ldots\to\pi_{i+1}SU(N)\to\pi_{i+4}SU(N)\to\pi_i H\to\pi_i SU(N)\to\ldots. \tag{9}$$

Let us assume $N=2$. Of the homotopic SU(2) groups, only the third contains the infinite cyclic group $\mathbf{Z}$. It follows from (9) that H must also have this property, i.e., must have only a single simple component which is locally isomorphic with SU(2). In this case, however, we are confronted by the contradiction in (9) at the value $i=6$, since we have $\mathrm{SU}(2)\approx S^3, \pi_7 S^3=\mathbf{Z}_2, \pi_{10}S^3=\mathbf{Z}_{15}, \pi_6 S^3=\mathbf{Z}_{12}$. In the SU(3) case, the only simple component of H could be SU(3), but this result again leads us to a contradiction in the case $i=6$. In the case of SU(4), it follows from (9) that $\pi_3 H$ contains $\mathbf{Z}+\mathbf{Z}$, while $\pi_{5,7}H$ contain $\mathbf{Z}$. Consequently, H has two simple components, and we need to consider the two versions $\mathrm{SU}(4)\times\mathrm{SU}(2)$ and $\mathrm{SU}(3)\times\mathrm{Spin}(5)$. Both cases lead to a contradiction at $i=8$. For the following values of N, we again first determine the infinite parts of the homotopic groups $\pi_i H$ dictated by (9). In this manner we can find simple components which might appear in H, since the infinite parts of the homotopic groups are known for all simple compact Lie groups. It turns out that for any $N\geqslant 5$ the only possible candidate is $\mathrm{SU}(N)\times\mathrm{SU}(N-2)$. In this case, however, we find a contradiction in (9) at $i=2N$. This contradiction can be established with the help of the data of Ref. 14 on unstable homotopic groups.

$$\pi_{2N}\mathrm{SU}(N)=\mathbf{Z}_{N!},\ \pi_{2N+1}\mathrm{SU}(N)=\begin{cases}0\\ \mathbf{Z}_2\end{cases},\ \pi_{2N+2}\mathrm{SU}(N)=\begin{cases}\mathbf{Z}_{(N+1)!/2} & \text{for even } N\\ \mathbf{Z}_2+\mathbf{Z}_{(N+1)!} & \text{for odd } N.\end{cases}$$

Let us assume, for example, that N is odd. The homotopic stratification sequence $\mathrm{SU}(N+1)/\mathrm{SU}(N)=S^{2N+1}$ gives us

$$0\to\pi_{2N+4}SU(N)\to\mathbf{Z}_2+\mathbf{Z}_{(N+2)!}\to\mathbf{Z}_{24}\,. \tag{10}$$

Consequently, $\pi_{2N+4}\mathrm{SU}(N)$ contains an element of order $\geqslant(N+2)!/24$, while the order of an arbitrary element $\pi_{2N}[\mathrm{SU}(N)\times\mathrm{SU}(N-2)]$ does not exceed $N!$ The monomorphism $\pi_{2N+4}\mathrm{SU}(N)\to\pi_{2N}H$ is therefore impossible.

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