Topology of gauge fields with several vacuums

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We consider topological invariants of gauge theories with several vacuums (metastable stage). A group-theoretical interpretation of the phases in ^{3}He is given.

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The existence of solutions of a monopole type[1,2] for equations of the Yang-Mills type interacting with a scalar Higgs field with SU(2) symmetry is connected with the presence of a topological charge. The formalism of homotopic groups turned out to be convenient for the calculation of this charge.

In theories with a higher symmetry group, other topological invariants arise, and make it possible to classify the obtained solutions.[1)] In this paper we analyze by topological methods the gauge equations with compact symmetry groups. In particular, we consider from this point of view the Ginzburg-Landau equation for ^{3}He.

1. We consider a Lagrangian of the Yang-Mills type with compact gauge group G

$$L = \tfrac{1}{2} \,|\, D_\mu \phi \,|^2 - \tfrac{1}{4} \mathrm{F}^2 - V(\phi), \tag{1}$$

where $D_\mu \phi$ is the covariant derivative:

$$D_\mu \phi = \partial_\mu \phi - ig\,\mathrm{AT}\,\phi \quad . \tag{2}$$

Here T is the isospin operator, A is the gauge field, and F is the curvature of the field A. The potential $V(\phi)$ is invariant to the action of the group G.

We asume that the field ϕ has a nonzero vacuum mean value (spontaneous breaking).

The manifold of the vacuums is defined as the set of the minima of the potential $V(\phi)$.

The gauge group G acts transitively on the vacuum manifold Θ. Consequently the manifold Θ can be represented in the form of a factor-space G/H, where H is the stationary group of the fixed vector $|\Psi\rangle$.[5]

2. It is known that several different vacuums can exist in a physical state. The concept of the "domain" structure of the vacuum was considered in a number of papers.[6] In our case we take "domain" to mean a region in Minkowski space or in Euclidean space R^n, and the field variables assume values in one of the vacuum manifolds. The fact that the components of the field can assume values in homogeneous spaces is in accord with the chirality of the theory.[7,8]

In the known one-dimensional (x,t) scalar theory with potential $V(\phi) = (-\mu^2/2)\phi^2 + 2\phi^4 (\lambda, \mu^2 > 0)$ there exist solutions (kinks, domain walls with asymptotic forms $\phi(x) = \mu/\sqrt{\lambda}\ (x \to +\infty)$, and $\phi(x) \to -\mu/\sqrt{\lambda}\ (x \to -\infty)$. The solution is of the form $\phi(x) = (\mu/\sqrt{\lambda})\tanh(\mu x/\sqrt{2})$.

From our point of view this means the existence of two vacuums in isotropic space. In this case the vacuums are the points $\mu/\sqrt{\lambda}$ and $-\mu/\sqrt{\lambda}$, and the solution itself with such asymptotic forms changes one vacuum into the other. In the general case, the vacuums themselves can be topologically nonequivalent.

Let R^3 be a physical space and let Ω_i be regions in R^3 with boundaries $\partial\Omega_i$. Following our physical analogy, we shall name the Ω_i domains.

There exist two types of mappings of domains in isotopic space J. The first type of mapping f is connected only with specification of the asymptotic values on the domain boundary $\partial\Omega_i$. For example, in solutions of the monopole type in an equation with SU(2) symmetry,[1,2] the mapping f is that of a sphere $S^2 \subset R^3$ into a sphere S^2 located in the isotopic space SU(2).

Another example, mapping of an entire domain, can be observe in ^{3}He. Here the mapping of the region filled by one of the phases $\Omega \subset R^3$ is given by an ordering matrix A_{pi} on one of the vacuums in isotropic space of complex 3×3 matrices.

In the general case we have n-dimensional regions $\Omega_1^n \cdots \Omega_k^n$ in R^n with boundaries $\partial\Omega_1^{n-1}$, $\partial\Omega_2^{n-1}, \cdots \partial\Omega_k^{n-1}$ which are $n-1$-dimensional smooth manifolds. The field ϕ determines the mapping f: $R^n \to J$, such that the boundaries $\partial\Omega_i$ go over into vacuum manifolds $V_1, \cdots V_e$

$$f : (\partial\Omega_1 \dots \partial\Omega_k) \to (V_1, \dots V_e). \tag{3}$$

The classification of these mappings makes it possible to establish the topological criteria of the existence of solutions of a definite type.

We consider an example when the manifold of vacuums is a product of two-dimensional spheres $S_1^2 \times S_2^2$. These vacuums are possible, for example, in the theory of O(4) symmetry. Let Ω_1 and Ω_2 be domains, let F be the mapping $R^3 \to J$, and the f_i the limitations F on $\partial\Omega_i$,f_i: $\partial\Omega_i \to S_1^2 \times S_2^2 (i = 1,2)$. Then there are no solutions that $\deg f_i = m^2 = 0$ on S_i^2 and $\deg f_i = 0$ on S_j^2 ($j \pm i$ $\deg f_j$ is the degree of the mapping).

The topological types of boundaries of vacuum manifolds can be quite diverse. For example, in the three-dimensional case some of them can be two-dimensional spheres, other can be toruses, etc.

3. We apply these general considerations to the Ginzburg-Landau theory of ^{3}He. From the mathematical point of view it can be regarded as a matrix field theory $A_{pi}(x)$ with Lagrangian[9] L

$$L = \sum_p \left\{ \frac{k_1}{2} \mid \operatorname{div} A_p \mid^2 + \frac{k_2}{2} \mid \operatorname{rot} A_p \mid + \frac{Q}{2} [A_p \operatorname{rot} A_p^* + A_p^* \operatorname{rot} A_p] \right\} + V(A), \tag{4}$$

where the potential $V(A)$ is given by

$$V(A) = \alpha Tr(AA^+) + \beta_1 \mid Tr(AA^T) \mid^2 + \beta_2 \mid Tr(AA^+) \mid^2 + \beta_3 Tr$$

$$\times \{ (AA^+)(AA^T)^* \} + \beta_4 Tr \{ (AA^+)^2 \} + \beta_5 Tr \{ (AA^+)(AA^+)^* \} . \tag{5}$$

$$(A_{ij}^T = A_{ji}, \ (A^*)_{ij} = A_{ij}^*, \ A^+ = A^{*T}.$$

The alternating fields are the components of the ordering function. It can be shown that there exist the following gauge transformations of the potential

$$A_{pi} = R_{pm} R_{in} e^{i\phi} A_{mn} \tag{6}$$

(summation over the repeated indices)

The matrices $R_{pm}, R_{in}, e^{i\phi}$ form a gauge group

$$H = SO_1(3) \times SO_2(3) \times U(1).$$

The structure of the group H is connected with the triplet structure of the superfluid liquid.

If follows from our analysis that the manifolds of the minima of the potential $V(A)$, corresponding to different phases in ^{3}He, can be only homogeneous spaces of group H. They take the following forms:

$$V_A = S^2 \times SO(3), \quad V_B = SO(3) \times U(1), \quad V_{A_1} = SO(3),$$

$$V_C = S^2 \times S^2 \times S^1, \quad V = S^2 \times S^2, \quad V = S^2 \times S^1.$$

The first two of them are the known A and B phases, and the third is the A_1 phase. The physical meaning of V_C is not clear.

We emphasize that from the group classification of the homogeneous spaces follows the listing of all the possible minima of the potentials $V(A)$, and consequently also of all the topological types of the vacuum manifolds—the phases of ^{3}He.

The topological conditions give only the necessary conditions for the existence of minima of the potential $V(A)$. Certain sufficient conditions of the existence of minima were obtained in[10].

From the physical point of view, the existence of topologically non-equivalent states in a system corresponds to an energy barrier between them. In the particular case of solutions of the monopole type, the question of the existence of an energy barrier between solutions with different topological charges was discussed earlier in[11].

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