Modified conservation laws for nonlinear waves

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(Submitted February 9, 1977)
Pis'ma Zh. Eksp. Teor. Fiz. **25**, No. 6, 296–299 (20 March 1977)

General expressions are obtained for the rate of change of the invariants of nonlinear waves under the influence of perturbations, under the assumption that in the absence of perturbations the corresponding equations can be solved by the inverse-problem method (the Korteweg–de Vries equations, the nonlinear Schrödinger equation, etc.).

PACS numbers: 03.40.Kf, 02.30.Jr

We assume that the system of equations for nonlinear waves can be written in the form

$$u_t(x,t) = S[u(x,t)] + \epsilon R[u(x,t)], \tag{1}$$

where S and R are nonlinear differential operators acting on $u(x,t)$ (u, generally speaking, can be complex and have several components) and Eq. (1) with $\epsilon = 0$ can be solved by the inverse-problem method (see[1–3], as well as a number of other papers, where this method is applied to different equations). Then, as is well known,[4,3] at $\epsilon = 0$ there exists an infinite number of polynomial conservation laws

$$\partial q_n[u,u^*]/\partial t + \partial p_n[u,u^*]/\partial x = 0 \quad (n = 1, 2, \ldots), \tag{2}$$

where q_n and p_n are polynomials of the functions u and u^* and of their spatial derivatives, so that the quantities $I_n\{u,u^*\} = \int_{-\infty}^{\infty} q_n dx$ are conserved if $u(x,t) \to 0$ as $|x| \to \infty$ (we confine ourselves only to these boundary conditions). The algorithm for the calculation of I_n is an inherent consequence of the method of the inverse problem[4,3]). If $\epsilon \neq 0$ in (1), then $dI_n/dt = \psi_n\{u,u^*\}$, where ψ_n are certain functionals of u and u^*. In this paper we obtain a general algorithm for the determination of $\psi_n(u,u^*)$.

The time derivative of the functional $I_n\{u,u^*\}$ can be written in the form

$$\frac{dI_n}{dt} = \int_{-\infty}^{\infty} \partial x \left[\frac{\delta I_n}{\delta u(x)} \frac{\partial u}{\partial t} + \frac{\delta I_n}{\delta u^*(x)} \frac{\partial u^*}{\partial t}\right], \tag{3}$$

where $\delta I_n/\delta u(x)$ is the functional derivative at the point x (if u is a real quantity then the last term of (3) vanishes). Substituting u_t and u_t^* from (1) in (3), we obtain

$$\frac{dI_n}{dt} = \int_{-\infty}^{\infty} \left[\frac{\delta I_n}{\delta u(x)} S + \frac{\delta I_n}{\delta u^*(x)} S^*\right] dx + \epsilon \int_{-\infty}^{\infty} \left[\frac{\delta I_n}{\delta u(x)} R + \frac{\delta I_n}{\delta u^*(x)} R^*\right] dx. \tag{4}$$

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The first integrand is a divergence. This can be verified directly (with the aid of rather cumbersome calculations). There is, however, a simple procedure, if it is noted that at $\epsilon = 0$ the derivative dI_n/dt must be an integral of a divergence. Consequently, under the considered boundary conditions we have

$$\frac{dI_n}{dt} = \epsilon \int_{-\infty}^{\infty} dx \left\{ \frac{\delta I_n}{\delta u\ (x)} R\,[u] + \frac{\delta I_n}{\delta u^*\,(x)} R^*\,[u] \right\} . \tag{5}$$

Thus, if the algorithm for writing down the invariants for Eq. (1) with $\epsilon = 0$ is known, then (5) makes it possible to calculate dI_n/dt due to a perturbation $\epsilon R[u]$. We shall call the relations (5) modified conservation laws.

If, for example, we use the representation $I_n = \int_{-\infty}^{\infty} q_n dx$, then

$$\frac{\delta I_n}{\delta u(x)} = \frac{\partial q_n}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial q_n}{\partial u_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial q_n}{\partial u_{xx}}\right) - \cdots , \tag{6}$$

and analogously for $\delta I_n/\delta u^*$. Frequently, however, it is more convenient to start from the relations between I_n and the Jost coefficients. We explain this using as an example the Korteweg—de Vries (KdV) equation, which corresponds to the eigenvalue problem

$$\frac{\partial^2 \phi}{\partial x^2} + [k^2 - u(x,t)]\,\phi = 0. \tag{7}$$

The eigenfunctions of this equation, for a continuous spectrum ($k^2 > 0$), can be chosen to be the Jost functions with asymptotic forms

$$f(x,k) \to e^{ikx}\,(x \to \infty), \quad g(x,k) \to e^{-ikx}(x \to -\infty),$$

which are connected with one another by the relations

$$g(x,k) = a(k) f^*(x,k) + b(k) f(x,k). \tag{8}$$

The Jost coefficients $a(k)$ and $b(k)$ are, just as I_n, functionals of u, with[5]

$$\ln a(k) = -\sum_{n=1}^{\infty} I_n/(2ik)^{2n-1}, \tag{9}$$

$$\frac{\delta a(k)}{\delta u(x)} = \frac{i}{2k} f(x,k)\, g(x,k). \tag{10}$$

Applying, in particular, these relations to the single-soliton state $u = u_s \equiv -2\kappa^2 \operatorname{sech}^2 z$, $z = \kappa(x - \xi)$ and recognized that in this case $b(k) = 0$, $a(k) = (k - i\kappa) \times (k + i\kappa)^{-1}$, $f(x,k) = \exp(ikx)(k + i\kappa)^{-1}$, we obtain

$$\left[\frac{\delta I_n}{\delta u(x)}\right]_{u=u_s} = \begin{cases} 1 & (n = 1) \\ 2^{2n-2}\kappa^{2n-2}\operatorname{sech}^2 z & (n > 1) \end{cases} . \tag{11}$$

Proceeding in similar fashion for the nonlinear Schrödinger equation, where $S[u]=i(u_{xx}/2+|u|^2u)$, $u_s=2\nu\,\mathrm{sech}z\exp(i\delta)$, and $z=2\nu(x-\xi)$, we obtain

$$\left[\frac{\delta I_{2m-1}}{\delta u(x)}\right]_{u=u_s}=\left[\frac{\delta I_{2m-1}}{\delta u^*(x)}\right]^*_{u=u_s}=(2\nu)^{2m-1}\,\mathrm{sech}\,z\,e^{-i\delta},$$

$$\left[\frac{\delta I_{2m}}{\delta u(x)}\right]_{u=u_s}=-\left[\frac{\delta I_{2m}}{\delta u^*(x)}\right]^*_{u=u_s}=-(2\nu)^{2m}\frac{\mathrm{th}\,z}{\mathrm{ch}\,z}e^{-i\delta}. \tag{12}$$

We can proceed in the same way also for n-soliton states. Relations (11) and (12) play an important role in the study of changes of the invariants of a soliton under the influence of perturbations. The last problem, however, is beyond the scope of the present article and will be considered separately.

The author is grateful to E. M. Maslov for useful discussions.

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