Finite-size effects and critical indices of 1D quantum models

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Critical indices which depend on continuous parameters are calculated for the case with finite-size effects.

1. A phase transition occurs at a zero temperature in quantum models in a 2D space-time, with a power-law long-wave asymptotic behavior of the correlation functions. This phase transition is described by a conformal field theory.[1,2] Finding the leading terms in the asymptotic behavior reduces to determining the central charge c in the corresponding representation of a Virassoro algebra and calculating the spectrum of anomalous dimensionalities Δ, $\overline{\Delta}$ of the operators in this representation. Finite-size effects[3,4] make it possible to find these quantities. In the case of periodic boundary conditions, the central charge c and the anomalous dimensionality of the operator ϕ are calculated from

$$E_L^0 = LE^0 - (\pi c/6v)/L + O(1/L^2), \tag{1}$$

$$E_L^\phi - E_L^0 = 2\pi v \theta_\phi/L; \quad P_L^\phi = (2\pi s_\phi/L) \pm P_\infty^\phi. \tag{2}$$

Here E_L^0 is the energy of the ground state $|0\rangle$ in a box of length L; E_L^ϕ is the minimum eigenvalue of the Hamiltonian (which corresponds to the eigenstate $|E_\phi\rangle$, for which the form factor is $\langle 0|\phi|E_\phi\rangle \neq 0$); P_L^ϕ is the momentum of this state, and v is the group velocity at the Fermi surface. The leading term in the long-wave asymptotic behavior of the correlation function of the fields ϕ, ϕ^* (* means charge conjugation) in the Euclidean formulation is

$$\langle \phi^*(z,\bar{z})\phi(0,0)\rangle = A \cos\{P_\infty^\phi(z+\bar{z})/2\}\,(z)^{-\Delta_\phi}(\bar{z})^{-\overline{\Delta}_\phi}, \tag{3}$$

where $\Delta_\phi = (\theta_\phi + s_\phi)/2$; $\overline{\Delta}_\phi = (\theta_\phi - s_\phi)/2$; $z = x + it$; and $\bar{z} = x - it$. These expressions differ from the standard expressions[3] in that it is not assumed that the gap in the spectrum of the momentum operator for the state $|E_\phi\rangle$ vanishes at $L = \infty$. It is also possible to calculate c from the low-temperature asymptotic behavior of the free-energy density in the limit[4,5] $L = \infty$:

$$f(T) = E^0 - (\pi c/6v)T^2 + O(T^2). \tag{4}$$

All of the critical indices can be found in this manner. This approach complements the construction method based on the general principles of conformal theory.[2,5,6] In the case $c < 1$, the set of critical indices is discrete and is determined unambiguously in the construction approach of Ref. 5. In the case $c \geqslant 1$, the critical indices may depend in a continuous way on the parameters of the theory. It is precisely this case in which expressions (1)–(4) are most effective.

2. Let us demonstrate how they work in integrable models, in the examples of a Bose gas and an XXZ spin-1/2 Heisenberg antiferromagnet. It follows from the low-temperature asymptotic behavior of the free energy[7,8] that we have $c=1$ in these cases. The corresponding Hamiltonians are written

$$H_{\rm BG}=\int_0^L dx\,[\partial_x\psi^+\partial_x\psi+\kappa\psi^+\psi^+\psi\psi-h\psi^+\psi],\ \kappa>0,\ h>0;$$

$$H_{XXZ}=\sum_{x=1}^{L}[\sigma_x^1\sigma_{x+1}^1+\sigma_x^2\sigma_{x+1}^2+\Delta\sigma_x^3\sigma_{x+1}^3+h\sigma_x^3/2],$$
$$-1<\Delta\leqslant 1;\ 0<h<4(1-\Delta).$$

Here h is the chemical potential of the Bose gas or the external magnetic field for the magnetic material. In the ground state in the Bose-gas model, pseudoparticles with a negative energy fill the Fermi band, and the Fermi momentum is $k_F=\pi N/L=\pi\rho$. In the limit $L\to\infty$, the density ρ remains finite. The ground state of the antiferromagnet can be characterized in an analogous way. In this case, the role of the length of the box is played by the number of lattice sites, and the "Fermi momentum" is determined by the strength of the magnetic field. The density is related to the magnetization by $M=(1-\rho)/2$.

For uncharged operators ($\phi_0=\psi^+\psi$ for the Bose gas and $\phi_0=\sigma^3$ for the magnetic material), those excited states $|E\rangle$ for which the relation $\langle 0|\phi_0|E\rangle\neq 0$ holds are formed by particles ($|k_p|>k_F$) and holes ($|k_h|<k_F$). The number of particles is equal to the number of holes. In the case $L=\infty$, there are two states with a minimum (0) energy: (a) $k_p=\pm k_F$, $k_h=\mp k_F$, $P_{\phi}^{\infty}=0$ and (b) $k_p=\pm k_F$, $k_h=\pm k_F$, $P_{\phi_0}^{\infty}=\pm 2k_F$. For the first state we easily find

$$E_L^{\phi_0,a}-E_L^0=2\pi v/L;\quad P_L^{\phi_0,a}=2\pi/L. \tag{5}$$

The second state has the lowest energy among those which are formed by a shift of the Fermi band as a whole ($\pm k_F\to\pm k_F+\delta$); here the energy shift is $\Delta E=Lv\delta^2$. The minimum possible value of δ is found from the Bethe equation: $\delta=2\pi z(\Lambda)/L$. Here $z(\lambda)$ is the "dressed charge," which is determined by a nonlinear integral equation (given in Ref. 9) and which has a simple physical meaning: $z(\lambda)=\partial\epsilon(\lambda)/\partial h$ [$\epsilon(\lambda)$ is the energy of an elementary excitation], where Λ is the value of the additive spectral parameter λ (Ref. 10) at the boundary of the Fermi band. We can then write

$$E_L^{\phi_0,b}-E_L^0=(2\pi v/L)\,2z^2(\Lambda);\quad P_L^{\phi_0,b}=\pm 2k_F. \tag{6}$$

From (5) and (6) we find the following expression for the simultaneous correlation function:

$$\langle\phi_0(x)\phi_0(0)\rangle\to(A/x^2)+B\cos(2k_Fx)/x^{\theta_0}+\langle\phi_0(0)\rangle^2, \tag{7}$$

$$\theta_0=2z^2(\Lambda). \tag{8}$$

The last term in (7) corresponds to vacuum-vacuum transitions. Expression (8) for the critical index reproduces the expression derived in Ref. 9. In the case of a repulsion (for a Bose gas and under the condition $\Delta>0$ in the magnetic material) we have $\theta_0>2$, and the nonoscillatory term is the leading term in (7). In the case of an attraction ($\Delta<0$), the oscillatory term is the leading term ($\theta_0<2$).

Let us examine the correlation function of the charged fields [$\phi_+(x) = \psi^+(x)$ or $\phi_+(x) = \sigma_x^+$]. The state $|E\rangle$ with the minimum energy (if $\langle E|\phi_+|0\rangle \neq 0$) is the ground state of the Hamiltonian in the sector with $N+1$ particles. We can then write $E_L^{\phi_+} - E_L^0 = (\partial h/\partial\rho)/L$, $P_L^{\phi_+} = 0$, and we find

$$\langle \phi_+^*(x)\phi_+(0)\rangle = c/x^{\theta_+}\,;\quad \theta_+ = (\partial\rho/\partial h)/2\pi v. \tag{9}$$

In integrable models we have $\partial\rho/\partial h = \pi v/z^2(\Lambda)$, and $\theta_+ = 1/\theta_0$. It is thus possible to prove the hypothesis offered in Ref. 9. The integrability of the theory was not utilized in the derivation of (9), so that the expression for θ_+ is of universal applicability. There is reason to believe that the relation $\theta_+ = 1/\theta_0$ will also be valid for nonintegrable models (e.g., Ref. 11).

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