Formation of solitary pulses in an amplifying nonlinear medium with dispersion

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A short solitary pulse is predicted to form in an extended amplifying medium with a negative dispersion and a nonlinearity of the refractive index at a slight saturation of the amplifying transition.

A study of solitons in dispersive media with a nonlinearity of the refractive index, which are described by a nonlinear Schrödinger equation, is of interest from both the general physical standpoint and the standpoint of applications. Such solitons were predicted in Refs. 1 and 2 and were observed experimentally by Mollenauer *et al.*[3] in 1980. No less interesting is a study of soliton-like pulse propagation regimes in amplifying media, in particular, in situations in which the amplification is nonlinear. As was shown in Ref. 4, in the case of linear amplification the amplitude of a quasisoliton increases without bound, its length decreases, and its energy increases exponentially. In a system of this sort, solitary pulses cannot exist. As we will show below, a solitary pulse may form in the case of a nonlinear amplification. Our purpose in the present letter is to study solitary pulses of this sort.

We write the polarization of the medium as

$$\mathcal{P} = \mathcal{P}^r + \mathcal{P}^n ,$$

where the resonant part of the polarization, associated with the amplifying centers, is $\mathcal{P}^r = \mathrm{Sp}(\hat{d}\hat{\sigma})$, and the operator $\hat{d}$ represents the dipole moment. The density matrix $\hat{\sigma}$ (in the interaction picture) is described by the equation

$$\frac{\partial \hat{\sigma}}{\partial t} + \hat{\Gamma}\hat{\sigma} = -\frac{i}{\hbar}[\hat{V}\hat{\sigma}], \tag{1}$$

where $\hat{\Gamma}$ is a relaxation operator, and $\hat{V} = -\hat{d}E$ is the Hamiltonian of the interaction of the atom with the field E. We assume that the populations of the resonant levels are changed only slightly by the field E. In this case, by working from (1), making use of the form of the relaxation operator $\hat{\Gamma}$ (given in Ref. 5), and using perturbation theory, we can easily find a solution $\hat{\sigma}$ and thus $\mathcal{P}^r$, in which we consider the nonlinearity of only lowest order. The nonresonant part of the polarization, $\mathcal{P}^n$, reflects the nonlinearity of the refractive index of the host and the linear loss in the medium. Substituting $\mathcal{P}$ into Maxwell's equation, and making use of the lowest-order dispersion, we finally find

$$i\frac{\partial q}{\partial \bar{z}} + \frac{\partial^2 q}{\partial \bar{t}^2} + 2q|q|^2 = -i\alpha q\int_{-\infty}^{\bar{t}} |q|^2 d\tilde{t} + i\beta q . \tag{2}$$

where we have introduced

$$q = C\sqrt{\pi n_2/2n_0}\ .$$

Here C is the slowly varying amplitude of the field E; n_2 is a measure of the increment in the refractive index proportional to $|E|^2$ $(n = n_0 + n_2|E|^2)$; and $\bar{z}$ and $\bar{t}$ are related to the longitudinal coordinate z and the time t by $\bar{z} = -z/\lambda$ and $\bar{t} = [-2/\lambda(\partial^2 k/\partial\omega^2)]^{1/2}[t - (\partial k/\partial\omega)z]$. The left side of Eq. (2) is the same as the well-known nonlinear Schrödinger equation. The first term on the right side stems from the slight saturation of the transition; the coefficient α is a measure of the "rate" at which the amplifying transition is saturated; the second term on the right side is a measure of the amplification in the medium; and β is the gain, which also incorporates the linear loss. In the derivation of (2), account was taken of the fact that the pulse length is much shorter than the relaxation time of the populations but much longer than the phase relaxation time of the amplifying transition.

A direct substitution easily verifies that Eq. (2) has a solution corresponding to a solitary wave:

$$q = \frac{\beta}{\alpha}\ \frac{\exp\{i[\xi_0\bar{t} + (\beta^2/\alpha^2 - \xi_0^2)\bar{z}]\}}{\operatorname{ch}\frac{\beta}{\alpha}(\bar{t} - 2\xi_0\bar{z} + \alpha\bar{z})}\ , \tag{3}$$

where ξ_0 is an arbitrary constant. It is easy to see that in the case $\alpha = \beta = 0$ expression (3) becomes a one-soliton solution of the nonlinear Schrödinger equation.[1,2] We can show that in this medium only solitary pulses of the type in (3) can form. Let us examine Eq. (2) in the case in which the coefficients α and β on the right side are small:

$$\alpha \to \epsilon\alpha, \quad \beta \to \epsilon\beta\ .$$

In this case it is useful to apply a formalism analogous to that developed in Ref. 4 for studying a perturbation of solitons of a nonlinear Schrödinger equation. We seek a solution of Eq. (2) in the form

$$q = \hat{q}(\theta, \zeta, \epsilon)\exp[i\bar{\xi}(\theta - \theta_0) + i(\sigma - \sigma_0)]\ , \tag{4}$$

where

$$\frac{\partial\theta}{\partial z} = -2\xi\ , \quad \frac{\partial\theta}{\partial\bar{t}} = 1, \quad \frac{\partial\sigma}{\partial\bar{z}} = \eta^2 + \xi^2, \qquad \frac{\partial\sigma}{\partial\bar{t}} = 0,$$

ξ, η, θ_0, and σ_0 are functions of the slow coordinate $\zeta = \epsilon\bar{z}$; and $\bar{\xi} = \xi + \epsilon(\partial\theta_0/\partial\zeta + \alpha)/2$. In the case $\epsilon = 0$ and $q = \eta\ \mathrm{sech}\ \eta(\theta - \theta_0)$, expression (4) is a one-soliton solution of the nonlinear Schrödinger equation with arbitrary constants[1] ξ, η, θ_0, and σ_0. Transforming to the new variables ζ and θ, and moving the small terms to the right side, we find from (2)

$$-\eta^2\hat{q}+\frac{\partial^2\hat{q}}{\partial\theta^2}+2\hat{q}|\hat{q}|^2=\epsilon\Big[-i\alpha\hat{q}\int_{-\infty}^{\theta}|\hat{q}|^2d\theta'+i\beta\hat{q}$$

$$-i\frac{\partial\hat{q}}{\partial\zeta}-i(\theta_{0\zeta}+\alpha)\frac{\partial\hat{q}}{\partial\theta}+(\theta-\theta_0)\hat{q}\frac{\partial\xi}{\partial\zeta}$$

$$-\hat{q}\Big(\xi\theta_{0\zeta}+\sigma_{0\zeta}-\frac{\epsilon}{4}[\theta_{0\zeta}+\alpha]^2\Big)\Big]=\epsilon\hat{F}(\hat{q}). \qquad (5)$$

We assume that $\hat{q}$ can be expanded in a series in ϵ:

$$\hat{q}(\theta,\zeta,\epsilon)=\hat{q}_0(\theta,\zeta)+\epsilon\hat{q}_1(\theta,\zeta)+\dots, \qquad (6)$$

where $\hat{q}_0=\eta\,\mathrm{sech}\,\eta(\theta-\theta_0)$. Using the notation $\hat{q}_1=\phi_1+i\psi_1$, and separating the real and imaginary parts in (5), we find, in first order in ϵ,

$$\hat{L}\phi_1=-\eta^2\phi_1+\frac{\partial^2\phi_1}{\partial\theta^2}+6\hat{q}_0\phi_1=\mathrm{Re}\hat{F}_1=(\theta-\theta_0)\hat{q}_0\frac{\partial\xi}{\partial\zeta}-(\xi\theta_{0\zeta}+\sigma_{0\zeta})\hat{q}_0, \qquad (7)$$

$$\hat{M}\psi_1=-\eta^2\psi_1+\frac{\partial^2\psi_1}{\partial\theta^2}+2\hat{q}_0^2\psi_1=\mathrm{Im}\hat{F}_1=-\alpha\hat{q}_0\int_{-\infty}^{\theta}\hat{q}_0^2d\theta'$$

$$+\beta\hat{q}_0-\hat{q}_{0\zeta}-(\theta_{0\zeta}+\alpha)\hat{q}_{0\theta}. \qquad (8)$$

It is not difficult to see that the operators $\hat{L}$ and $\hat{M}$ are self-adjoint. Making use of the identities $\hat{L}\hat{q}_{0\theta}\equiv0$ and $\hat{M}\hat{q}_0\equiv0$, and noting that ϕ_1 and ψ_1 are localized along θ, we find

$$\int_{-\infty}^{\infty}\hat{q}_{0\theta}\,\mathrm{Re}\hat{F}_1\,d\theta=0, \qquad \int_{-\infty}^{\infty}\hat{q}_0\,\mathrm{Im}\,\hat{F}_1\,d\theta=0. \qquad (9)$$

From the first condition in (9) we find $\partial\xi/\partial\zeta=0$; from the second we find the dependence of the amplitude η on ζ:

$$\eta=\frac{\beta}{\alpha}\,\frac{\eta_0e^{2\beta\zeta}}{\eta_0(e^{2\beta\zeta}-1)+\beta/\alpha}. \qquad (10)$$

It follows from this result that the pulse amplitude η tends toward β/α in the limit $\beta\zeta\to\infty$. Integrating Eqs. (7) and (8), we find

$$\phi_1=-\frac{1}{2\eta}(\xi\theta_{0\zeta}+\sigma_{0\zeta})[1-\eta(\theta-\theta_0)\mathrm{th}\,\eta(\theta-\theta_0)]\,\mathrm{sech}\,\eta(\theta-\theta_0),$$

$$\psi_1=-\frac{\eta}{2}(\beta-\alpha\eta)(\theta-\theta_0)^2\,\mathrm{sech}\,\eta(\theta-\theta_0). \qquad (11)$$

Quasisteady solutions (11) hold in the region

$$\eta\,|\theta-\theta_0| \ll \min\left[\frac{1}{\sqrt{\epsilon}}\left(\frac{2\eta}{|\beta-\alpha\eta|}\right)^{1/2},\ \frac{1}{\epsilon}\frac{2\eta}{|\xi\theta_{0\zeta}+\sigma_{0\zeta}|}\right] \tag{12}$$

To determine the behavior of the parameter $\xi\theta_{0\zeta}+\sigma_{0\zeta}$, we make use of an integral of motion—the energy of the pulse—which can be found from Eq. (2):

$$\frac{\partial}{\partial\zeta}\int_{-\infty}^{\infty}|q|^2\,d\tilde{t} = 2\int_{-\infty}^{\infty}|q|^2\,dt\left(\beta-\frac{\alpha}{2}\int_{-\infty}^{\infty}|q|^2\,dt\right). \tag{13}$$

Substituting $\hat{q}$ from (6) into this relation, we find, in first order in ϵ,

$$\int_{-\infty}^{\infty}\hat{q}_0\phi_1\,d\theta = \int_{-\infty}^{\infty}(\hat{q}_0\phi_1)\Big|_{\zeta=0}\,d\theta\,\frac{\beta^2}{\alpha^2}\,e^{2\beta\zeta}\left[\eta_0(e^{2\beta\zeta}-1)+\frac{\beta}{\alpha}\right]^{-2}, \tag{14}$$

which tells us that

$$\xi\theta_{0\zeta}+\sigma_{0\zeta} = (\xi\theta_{0\zeta}+\sigma_{0\zeta})\Big|_{\zeta=0}\,\frac{\beta^2}{\alpha^2}\,e^{2\beta\zeta}\left[\eta_0(e^{2\beta\zeta}-1)+\frac{\beta}{\alpha}\right]^{-2} \tag{15}$$

tends toward zero with increasing ζ. To determine the motion of the parameter, we make use of the ζ dependence of the correction to the energy of second order in ϵ. From (13) we find, in second order,

$$\int_{-\infty}^{\infty}(2\hat{q}_0\phi_2+\psi_1^2)\,d\theta = \int_{-\infty}^{\infty}(2\hat{q}_0\phi_2+\psi_1^2)\Big|_{\zeta=0}\,d\theta\,\exp 2\left(\beta-2\alpha\int_0^{\zeta}\eta\,d\zeta'\right), \tag{16}$$

where, as follows from (18), ϕ_2 is determined by the following equation in second order:

$$\begin{aligned}\hat{L}\phi_2 &= -\eta^2\phi_2+\frac{\partial^2\phi_2}{\partial\theta^2}+6\hat{q}_0^2\,\phi_2\\ &= -2\hat{q}_0(\psi_1^2+3\phi_1^2)+\alpha\psi_1\int_{-\infty}^{\theta}\hat{q}_0^2\,d\theta'-\beta\psi_1+\frac{\partial\psi_1}{\partial\zeta}\\ &+(\theta_{0\zeta}+\alpha)\frac{\partial\psi_1}{\partial\theta}+\frac{1}{2}(\theta-\theta_0)\hat{q}_0\,\theta_{0\zeta\zeta}-q_0(\theta_{0\zeta}^2-\alpha^2) = \mathrm{Re}\hat{F}_2\,.\end{aligned} \tag{17}$$

Analysis of Eq. (17) and also of the equations for $\hat{q}_j$ of higher orders shows that quasisteady solutions are definitely valid in the region[1)]

$$\eta\,|\theta-\theta_0| \ll \frac{1}{2}\ln\frac{1}{\epsilon}$$

At large values of ζ, we can ignore terms proportional to ψ_1 and ϕ_1 on the right side of (17). Carrying out the integration in (17), and substituting ϕ_2 into (16), we find

$$\theta_{0\zeta}^2-\alpha^2 = (\theta_{0\zeta}^2-\alpha^2)\Big|_{\beta\zeta_0\gg 1}\,\frac{\beta^2}{\alpha^2}\,e^{2\beta\zeta}\left|\eta_0(e^{2\beta\zeta}-1)+\frac{\beta}{\alpha}\right|^{-2}, \tag{18}$$

from which it follows that we have $\theta_{0\xi} \to \pm\alpha$ in the limit $\beta\xi \to \infty$. Noting that $\theta_{0\xi}$ generates a correction to the pulse propagation velocity, we find that there are two values of the propagation velocity for a solitary pulse (for a given value of ξ):

$$V_{\pm}^{-1} = k' - \xi\sqrt{-\frac{2k''}{\lambda}} \pm \alpha\epsilon\sqrt{-\frac{k''\lambda}{2}} = V_c \pm \alpha\epsilon\sqrt{-\frac{k''\lambda}{2}} .$$

Either of these values may be reached, depending on the initial value $\theta_{0\xi}(\zeta_0)$.

We thus find that at large values of $\beta\xi$ the corrections ϕ_1 and ψ_1, like the corrections of higher orders [see (10), (15), and (18)], tend toward zero, that we have $\eta \to \beta/\alpha$, and that an arbitrary distribution of the type in (4) converts into a solitary pulse as in (3). Consequently, a solitary secant pulse (3) does in fact form in the medium. Estimates for typical parameter values of single-mode fiber lightguides activated by Nd^{3+} ions, with an ion inversion density of 10^{19} cm^{-3}, show that for a gain value lying between 0.27×10^{-3} and 0.54×10^{-3} cm^{-1} the energy density of a solitary pulse as in (3) lies between 0.34 and 0.68 mJ/cm^2 (depending on the gain), and the range of the pulse length τ_s is correspondingly $\tau_s = 2\alpha\beta^{-1}(-k''\lambda/2)^{-1/2} \approx 2$–20 ps.

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[1)]*Calculations*[6] *show that outside this region the condition for a quasisteady nature of the solutions is* violated. Nevertheless, under the condition $\eta|\theta - \theta_0| \gtrsim \frac{1}{2}\ln(\epsilon^{-1})$ a solution can be found, since the nonlinear term on the right side of (5) is small in this case $(2\hat{q}|\hat{q}|^2 \lesssim \epsilon)$ and can be treated as a perturbation.

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