Evolution of perturbations in an inflationary universe

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The problem of the behavior of scalar perturbations of the metric of the universe in a theory of gravitation with higher derivatives is solved. Inhomogeneities generated from quantum fluctuations may be sufficient for the formation of galaxies. Observational constraints on the parameters of the theory are found.

There are several reasons for the considerable interest in theories of gravitation with higher derivatives. First, non-Einsteinian increments in the effective action due to the presence of higher derivatives may result from, for example, effects of the polarization of vacuum of physical fields in an external gravitational field or a string theory.[9] Futhermore, there are grounds for expecting that it would be possible to construct a satisfactory renormalizable quantum theory of gravitation on the basis of theories with higher derivatives.[10]

In its early stages of evolution, the universe may have gone through an inflationary stage because of an effective cosmological constant which arises from increments in the Einstein equations which are nonlinear in the curvature, as was shown in Refs. 7 and 8. In inflationary models there is the possibility in principle of explaining the origin of the nucleating inhomogenities which are required for the formation of galaxies.[1–4] Because of the presence of the higher derivatives, there is no basis for expecting at the outset that the picture of the evolution of perturbations in these theories will be reminiscent of the evolution of perturbations in qualitatively different models of inflation with a scalar field. Quantum fluctuations of the metric against the background of a de Sitter universe were examined in a model with higher derivatives in Ref. 2. In the present paper we analyze the behavior of perturbations in various stages of the evolution of the universe.

We consider a theory with a total action

$$S= \frac{1}{16\pi G} \int d^4x\sqrt{-g}\,(-R + \frac{1}{6M^2} R^2). \qquad (1)$$

A further analysis shows that the results derived for perturbations in this model remain qualitatively valid for a broad range of theories with higher derivatives.

As in Ref. 4, it is convenient to use the metric of a homogeneous and isotropic plane cosmological model with small scalar perturbations in the relativistic potential gauge[5]:

$$ds^2 = a^2(\eta)[(1 + 2\phi)d\eta^2 - (1 - 2\psi)\delta_{\alpha\beta}dx^\alpha dx^\beta]. \qquad (2)$$

We need the following equations for the unperturbed model which follow from Einstein's equations:

$$R'' + 2\alpha R' + M^2a^2R = 0, \tag{3}$$

$$\alpha^2 - \alpha' = \frac{F''}{2F} - \alpha\frac{F'}{F}. \tag{4}$$

The prime means differentiation with respect to the conformal time η:$\alpha = a'/a$: and $F = 1 - R/3M^2$. From the $0-0$, $0-\alpha$ and $\alpha-\beta$ ($\alpha\neq\beta$) of the Einstein's equations we find the following respective equations for perturbations:

$$\Delta\psi - 3\alpha\psi' - 3\alpha^2\phi = \frac{1}{2M^2F}\left[\alpha\delta R' - \frac{1}{3}\Delta\delta R - \alpha'\delta R - 2\alpha R'\phi - R'\psi'\right], \tag{5}$$

$$\psi' + \alpha\phi = \frac{1}{6M^2F}\left[R'\phi + \alpha\delta R - \delta R'\right], \tag{6}$$

$$\delta R = 3M^2F(\phi - \psi). \tag{7}$$

Using (4), we can reduce the solution of system (5)–(7) to the solution of the following second-order equation for the variable $u = F^{3/2}a(\phi + \psi)/F'$:

$$u'' - \Delta u - \frac{z''}{z}u = 0;\quad z = \left(\frac{(a\sqrt{F})'}{a^2F'}\right). \tag{8}$$

In turn, the solution of this equation is easily found in asymptotic cases. Using the plane-wave perturbation $u \propto e^{ikx}$ along with (6) and (7), we find ϕ and ψ from (8). For long-wave perturbations with $k^2 \ll z''/z$ we find

$$\phi = C\left(\frac{1}{aF}\int aFdt\right)^{\bullet},\quad \psi = \phi + C\frac{\dot{F}}{aF^2}\int aFdt, \tag{9}$$

where $t = \int a d\eta$, and the superior dot means differentiation with respect to t. These expressions can also be derived by the methods of Ref. 6, which are valid only for homogeneous perturbation modes. For short-wave pertrubations with $k^2 \gg z''/z$ we find

$$\phi = -\frac{1}{3F^{1/2}}\left[\left(\frac{\ddot{F}}{\dot{F}} - \frac{5}{2}\frac{\dot{F}}{F} + \frac{\dot{a}}{a}\right)\left(C_1\sin\left(k\int\frac{dt}{a}\right)\right.\right.$$

$$\left.\left.+ C_2\cos\left(k\int\frac{dt}{a}\right)\right) + \frac{k}{a}\left(C_1\cos\left(k\int\frac{dt}{a}\right) - C_2\sin\left(k\int\frac{dt}{a}\right)\right)\right],$$

$$\psi = -\phi + \frac{\dot{F}}{F^{3/2}}\left(C_1\sin\left(k\int\frac{dt}{a}\right) + C_2\cos\left(k\int\frac{dt}{a}\right)\right). \tag{10}$$

The asymptotic quasi–de Sitter evolution regime is known to apply to a wide range of solutions of Eqs. (3) and (4) for an unperturbed background model[7]:

$$a \propto e^{\int H(t)dt}, \quad H = \frac{M^2}{6}(t_s - t), \tag{11}$$

where $\dot{H} \ll H^2$. Substituting (11) into (9), we find the time dependence of the amplitude of a nondecaying mode of long-wave perturbations in the quasi–de Sitter stage:

$$\phi \cong C\left(\frac{1}{H}\right)^{\bullet} \cong -C\,\frac{\dot{H}}{H^2}, \quad \psi \cong -\phi. \tag{12}$$

In the last, scalaron, stage we have[8]

$$a \propto t^{2/3}\left[1 + \frac{2}{3Mt}\sin Mt\right] \tag{13}$$

and, correspondingly,

$$\phi \cong \frac{3}{5}C, \quad \psi = \phi. \tag{14}$$

It follows from (12) and (14) that from the time at which the long-wave regime is reached, the perturbation amplitude increases by a factor of $\frac{3}{5}(H^2/\dot{H}) \cong 3.6(H^2/M^2)$. We also note that perturbations which are conformally planar in the quasi–de Sitter stage ($\phi \cong -\psi$) convert into conformally Newtonian perturbations ($\phi \cong \psi$) at the transition to the scalaron stage. This circumstance distinguishes their behavior from that of perturbations of metric in the case of a scalar field, in which we always have $\phi = \psi$. A more detailed analysis of (9) shows that oscillatory corrections to (14) arise in stage (11). In contrast with the case of a scalar field, the oscillatory parts of the perturbations of the metric, ϕ and ψ, are in this case displaced with respect to each other by a half-period. This circumstance is important for analyzing the subsequent decay of scalarons into other particles.

With regard to short-wave perturbations, we find from (10) that for sufficiently large values of k such perturbations are conformally planar ($\phi \cong -\psi$) in all stages of the evolution of the universe.

From the results derived here and from limitations on the amplitude of perturbations at the galactic scale ($\phi = \psi < 10^{-4}$) we find limitations on M and H. Using $\phi = -\psi = Ml$ at the time of the transition to the long-wave regime,[2] and taking into account the gain of the following amplification of the perturbations ($3.6H^2/M^2$), we find $M < 10^{+13}$ GeV and $H < 5 \times 10^{13}$ GeV, where H is the value of the Hubble constant in the quasi–de Sitter stage, at the time at which the long-wave asymptotic behavior sets in for those perturbations which are responsible for the formation of the structure of the universe.

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