Orbital dynamics in ^{3}He-A and effective action

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The dynamic orbital angular momentum L_0 in ^{3}He-A is derived from the effective action of fermions in an external field. The result $L_0 = 1/2(\rho - C_0)$ is shown to hold in all orders in Δ_0/ϵ_F. The relationship with the chiral anomaly in quantum electrodynamics is studied. The correct regularization in the fermion action in the case of ^{3}He-A is pointed out.

It has recently been shown[1–4] that field-theory methods can be used successfully to describe ^{3}He-A. On the one hand, the Bogolyubov equations for ^{3}He-A in the presence of a texture are analogous to the Dirac equations in an external field. On the other, the vanishing of the energy gap in the quasiparticle spectrum at two points on the Fermi surface, at $\mathbf{k} = \pm k_F \mathbf{l}$, leads to the existence of massless fermions. The anomalies for massless fermions which are produced by gauge fields and which are quite familiar in field theory[5,6] also exist in ^{3}He-A. This circumstance makes it possible to resolve several paradoxes in the theory of superfluid ^{3}He-A (Refs. 2–4).

There is, however, a substantial difference between the anomalous properties in quantum electrodynamics and those in ^{3}He-A, because of the difference in the deep vacuum levels of fermions, where the Bogolyubov equations differ from the corresponding dirac equations. In particular, the anomaly which stems from the nonconservation of the chiral current, $\partial_\mu J^\mu = e^2/16\pi^2 F^*_{\mu\nu} F^{\mu\nu}$, i.e., from the pumping of momentum from vacuum to excitations, has identical forms in the two theories, since it is determined by fermion levels near $\pm k_F \mathbf{l}$; however, the vacuum-current structure itself, which is determined by deep vacuum levels, is markedly different in the two cases. This point was demonstrated in Refs. 2 and 3, where the time-dependent component of J^0, which plays the role of a mass flux in ^{3}He-A, was calculated. In the present letter we calculate the spontaneous orbital angular momentum in ^{3}He-A, i.e., the current components J^μ which are transverse with respect to $\mathbf{l}$. In quantum electrodynamics these components correspond to spatial components of the current.[7,8] We will show that the anomalously small value of the spontaneous orbital angular momentum in comparison with its analog in quantum electrodynamics ($L_0 \cong [\hbar\rho(\Delta_0/\epsilon_F)^2 \ln(\epsilon_F/\Delta_0)] \sim \hbar\rho \times 10^{-6}$) is a consequence of the difference between the Bogolyubov operator in ^{3}He-A and the Dirac operator.

The effective action describing the dynamics of the vector $\mathbf{l}$ is[9]

$$S_{eff} = -\ln \det M, \tag{1}$$

where

$$\begin{Bmatrix} \epsilon - i\partial_t;\ \alpha p_i(\Delta_1^i + i\,\Delta_2^i) \\ \alpha p_i(\Delta_1^i - i\,\Delta_2^i);\ -\epsilon - i\partial_t \end{Bmatrix};$$

$\vec{\Delta}_1, \vec{\Delta}_2, [\vec{\Delta}_1 \vec{\Delta}_2] = \mathbf{l}$ are the unit vectors of the orbital coordinate system; the spin part of the order parameter is assumed constant; Δ_0 is the gap amplitude; $\alpha = \Delta_0/k_F$; and $\epsilon = (p^2/2) - \mu$ is the energy reckoned from the Fermi level. Ignoring the derivatives of the order parameter, and choosing $\mathbf{l} = \mathbf{l}(0) + \delta\mathbf{l}(r)$, where $\mathbf{l}(0) \| \hat{\mathbf{z}}$, we can write M as

$$M = \{\tau_3 \partial_t + \alpha(\tau_1 P_1 + \tau_2 P_2) + i\epsilon\} = \not{p} + i\epsilon, \tag{2}$$

where the τ_i are the Pauli matrices, and $P_i = \hat{p}_i - \delta l_i \hat{p}_z$. (Ref. 2). Near the poles of the Fermi sphere, expression (2) becomes the Dirac operator in an external field[1-3] $\mathbf{A} = k_F \delta\mathbf{l}$. The operator M may be thought of as the Dirac operator in 2 + 1 dimensions in an external field with a mass ϵ, which is a quadratic operator. It has been established elsewhere that an anomaly of parity $j_\mu \sim \epsilon_{\mu\nu\alpha} F_{\nu\alpha}$ exists in a 2 + 1 quantum electrodynamics.[10,11] We will show that the orbital-angular-momentum paradox is a manifestation of the same anomaly in ^{3}He-A. For this purpose, we examine the response of the action $S_{\rm eff}$ to an adiabatic change in $\delta\mathbf{l}_\perp$, e.g., $\delta\mathbf{l}_x$. In this case we find from (1) and (2)

$$\frac{\delta S_{eff}}{\delta l_x} = \frac{1}{2} \mathrm{Tr} \frac{\alpha \tau_1 \hat{p}_z}{\not{p} + i\epsilon} = \frac{1}{2} \int_{-\infty}^{0} dS \, \mathrm{Tr}\, e^{S(\not{p}^2 + \epsilon^2)} \alpha \tau_1 \hat{p}_z (\not{p} - i\epsilon)$$

$$= \frac{1}{2} \int_{-\infty}^{0} dS \sum_{k_\mu} e^{-ik_\mu x_\mu} \, \mathrm{tr}\, e^{S(\not{p}^2 + \epsilon^2)} \alpha \tau_1 \hat{p}_z (\not{p} - i\epsilon) e^{ik_\mu x_\mu}, \tag{3}$$

where we have used the standard procedure[5,11] for calculating the Tr. As a result, taking two spin projections into account, we find the following results from (3) in the leading order in the gradients:

$$\frac{\delta S_{eff}}{\delta l_x} = -L_0 \partial_t l_y, \quad L_0 = -\frac{\alpha^2}{4} \sum_{k,S} k_z^2 \frac{\partial}{\partial \epsilon} \frac{1}{E}, \tag{4}$$

where $E = \sqrt{\alpha^2(p_x^2 + p_y^2) + \epsilon^2}$ is the energy of the quasiparticles. The quantity L_0 is none other than the spontaneous orbital angular momentum in[7,8] ^{3}He-A. Integrating the expression for L_0 by parts, we find the *exact* formula[1)]

$$L_0 = \frac{1}{2} (\rho - C_0), \tag{5}$$

where $\rho = \Sigma_k n_k = \frac{1}{2} \Sigma_k (E - \epsilon/E)$ is the density of particles, and $C_0 = k_F^3/3\pi^2$, in complete agreement with the phenomenological description of Ref. 12, which uses the algebra of Poisson brackets.

In (3) we have taken into account that part of the action which is linear in the time derivative and which was omitted from Ref. 9, where only the terms quadratic in $\mathbf{l}$ and $\nabla_i \mathbf{l}$ in the action S_2 were derived. Taking these terms into account, we find the following equations of the orbital dynamics:

$$\frac{\delta S_2}{\delta \mathbf{l}} = -\frac{1}{2} (\rho - C_0) \left[\mathbf{l} \frac{\partial \mathbf{l}}{\partial t} \right]. \tag{6}$$

There is a relationship betwen L_0 and the chiral current $\mathbf{J}$ in quantum electrodynamics. The final result for the effective action depends on the regularization method. If we restrict the analysis to the linear expansion, $\epsilon \cong (p_z - k_F) V_F$, and introduce a cutoff which is symmetric with respect to ϵ_F, we find from (3) the chiral vacuum current $\mathbf{J}_{\rm vac} = (\delta S_{\rm eff}/\delta \mathbf{A}) = e^2/4\pi^2[\mathbf{A}\mathbf{E}]$, where $\mathbf{E} = k_F \dot{\mathbf{l}}$, is the electric field.[3] If we instead treat ϵ as a quadratic operator, in accordance with ^{3}He-A, we find the result in (6) for the spontaneous orbital angular momentum. The magnitude of the orbital angular momentum of vacuum, $L^{\rm vac} = \hbar\rho/2$, is thus renormalized substantially when account is taken of the angular momentum, i.e., the spatial components of the current of excitations, for which the Dirac-equation approximation is valid. Furthermore, it follows from (6) that there is no four-dimensional Lagrangian whose variation would lead to Eq. (6); this is a standard situation encountered with the Landau-Lifshitz equations. Consequently, to correctly describe the orbital dynamics in ^{3}He-A, we need to introduce a Wess-Zumino action.[4] In this approach, the Wess-Zumino action is found with the correct coefficient, and, in principle, it can lead to an unusual statistics of vortices.[13]

We thus introduce an additional parameter τ; $\mathbf{l}$ depends on τ. In this case, by analogy with the derivation of (3)–(6), we find, in the adiabatic approximation,

$$\delta S_{eff} = \int d\tau \frac{\partial \mathbf{l}}{\partial \tau} \frac{\delta S_{eff}}{\delta \mathbf{l}} = \int d^3x dt d\tau \frac{1}{2} (\rho - C_0) \mathbf{l} \left[\frac{\partial \mathbf{l}}{\partial t} \frac{\partial \mathbf{l}}{\partial \tau} \right]. \tag{7}$$

Expression (7) is the Wess-Zumino action. If we take account of the phase of the order parameter, ϕ, and the combined symmetry in ^{3}He-A, $U^1_{\rm comb} = \{\phi \to \phi + \alpha, \vec{\theta} \to \vec{\theta} - \alpha \mathbf{l}\}$, where $\vec{\theta}$ is the angle through which the order parameter is rotated, we can show that the total action is

$$S_{eff} = \int d^3x dt \frac{\rho}{2} \left(\frac{\partial \phi}{\partial t} + \frac{\partial \vec{\theta}}{\partial t} \mathbf{l} \right) - \frac{1}{2} \int C_0 \mathbf{l} \left[\frac{\partial \mathbf{l}}{\partial t} \frac{\partial \mathbf{l}}{\partial \tau} \right] d^3x dt d\tau. \tag{8}$$

It follows from (8) that C_0 is dynamically invariant and that the angular momentum is quantized: $\int d^3_x L_0 = N/2$, where N is an integer.[4,12] We also note that the second term in (8) may be thought of as a Berry phase, since as $\mathbf{l}$ varies adiabatically along a closed contour on a sphere, this term is equal to the area swept out by the vector $\mathbf{l}$, i.e., to the solid angle which rests on this contour.[14]

The paradox of the spontaneous orbital angular momentum is thus resolved in a natural way when we consider how the quasiparticles which are interacting with the boson field of the vector $\mathbf{l}$ affect the dynamics of the orbital-angular-momentum vector. The small value of the spontaneous orbital angular momentum in superfluid ^{3}He-A is attributed to the small coefficient α in the interaction of the quasiparticles with the field $\mathbf{l}$. Furthermore, the phenomenological relationship among ρ, C_0, and L_0 found previously[12] is verifed completely. The expression for L_0 agrees with the static angular momentum, which is also small,[15] on the order of $(\Delta_0/\epsilon_F)^2$.

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