

Ground state and Josephson dynamics of the double-well condensates interacting with trap oscillations

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In the present paper it is shown that the interaction between classical anharmonic oscillations of a trapped condensate and excited Josephson states corresponding to a large enough initial disbalance of the particle number generates their equilibrium bound state. The dynamics of the system is determined by the self-consistent oscillations of the initial disbalance of the particle number and condensate shape. The existence of the bound state implies that the Josephson states can be detected by observing the change of the condensate shape.

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1. Since the discovery of two-component condensates [1] as well as condensates in a double-well potential [2] the phenomena caused by phase coherence of two condensate modes attract considerable interest, both experimental and theoretical (see, e.g., [3] and references therein). In the work [2] spatial quantum coherence was observed by means of interference pattern in two overlapping condensates. In the work [4] coherent oscillations of the relative populations have been observed in driven two-component condensates with different internal states. As is well known, a bright manifestation of phase coherence is Josephson effect. In numerous studies devoted to the Josephson effect in systems of two condensates in different internal states [4, 5] or in a double-well potential [6, 7] coherent Josephson oscillations are considered for various dynamical regimes caused by the competition between tunneling and intracondensate interaction (nonlinearity). However, experimental observation of the Josephson effect is difficult because the small energy splitting associated with Josephson coupling means that thermal and quantum fluctuations will destroy the phase coherence between two condensates even at the lowest achievable temperatures [8, 9]. While the energy splitting can be increased, for instance, by lowering of the barrier height, it then becomes comparable with that of motion states of the condensates.

The problem of interaction between the degrees of freedom generated by the Josephson coupling and states of motion (oscillations) of the trapped condensates has been partially analyzed in the work [10]. The present paper focuses on a mechanism of formation of the bound state of the Josephson degrees of freedom and trap os-

cillations due to their interaction. The mechanism proposed in what follows may be important for detection of the excited Josephson states.

As is well known [3, 6, 7, 12] the Hamiltonian of two condensates with Josephson coupling has the form

$$H_J - E_0 = E_C(\Delta N)^2 - 2E_J \cos \phi, \quad (1)$$

here $\Delta N = N_1 - N_2$, $E_C = \partial\mu/\partial N$, $\mu \equiv \mu_1 = \mu_2$ are the chemical potentials, N are the total particle numbers, E_J describes the Josephson (tunneling) coupling, $\phi = \theta_1 - \theta_2$ is the relative phase of the condensates. The quantities E_C , E_J depend on the total particle number N . (ΔN) , ϕ are canonically conjugate variables. In (1) the energy origin is the mean-field summary energy of the condensates, namely, $E_0 \equiv \mu N$. The quantization of the Hamiltonian (1) produces the spectrum of the particle number in the Josephson potential $E_J \cos \phi$. The states of this spectrum represent the quantum analogue of Josephson oscillations. As is shown in what follows, any Josephson state can be realized by means of a given initial disbalance of the particle number. For this reason, it is interesting to obtain the complete spectrum generated by the Hamiltonian of (1), and to represent it as a function of the initial disbalance. The Schrodinger equation for the Hamiltonian (1) is derived by means of the quantization rule: $(\Delta N) \rightarrow -i\partial/\partial\phi$. As a result, we obtain Mathieu's equation. The spectrum for this equation has been derived in the work [11]. Here we obtain the relation between the quantum numbers and initial disbalance of the particle number. In the Josephson regime $\Delta N \gg 1$ [3], the discrete spectrum within the range of $-2E_J < \varepsilon < 2E_J$ is specified by the Bohr-Sommerfeld formula:

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$$\begin{aligned} \nu(\varepsilon_\nu) &= \oint \frac{d\phi_r}{\pi} \Delta N(\phi_r; \varepsilon_\nu) = \\ &= \oint \frac{d\phi_r}{\pi} \left[\frac{1}{E_C} (\varepsilon_\nu + 2E_J \cos \phi) \right]^{1/2} = \\ &= \nu_c [E(\kappa) - (1 - \kappa^2)K(\kappa)], \end{aligned} \quad (2)$$

$$\nu_c = \frac{8}{\pi} \left(\frac{E_J}{E_C} \right)^{1/2}, \quad \kappa^2 = \frac{\varepsilon_\nu + 2E_J}{4E_J}, \quad \nu_c \gg 1;$$

$K(\kappa)$ and $E(\kappa)$ are complete elliptic integrals of the first and second kinds. The quantity ν_c is of the maximum possible value of the number of levels in the well. In what follows, the states with $\nu < \nu_c$ are denoted as "libration" states. In the region of $\nu \geq \nu_c$ the $\nu(\varepsilon)$ dependence is determined by the following expression

$$\nu(\varepsilon) = \frac{4}{\pi^2} \left(\frac{E_J}{E_C} \right)^{1/2} \kappa E(\kappa^{-1}). \quad (3)$$

The states (3) with $\nu > \nu_c$ are the self-trapping states [7]. Eqs.(2), (3) imply that

$$\begin{aligned} \varepsilon(\nu) &\approx -2E_J + \omega_m \nu \quad \text{at } 1 \ll \nu \ll \nu_c, \\ \varepsilon(\nu) &\approx 4\pi^2 E_C \nu^2 \quad \text{at } \nu \gg \nu_c. \end{aligned} \quad (4)$$

At the same time, it is easy to show that $d^2\varepsilon(\nu)/d\nu^2 < 0$ at $\nu < \nu_c$ and $d^2\varepsilon(\nu)/d\nu^2 > 0$ at $\nu > \nu_c$. At $\nu = \nu_c$ the curve $\varepsilon(\nu)$ has a flex point.

Since the energy is conserved, the state with a given ν value can be realized by means of the definition of the initial values of $(\Delta N)_0$ and $\phi(0)$. Namely, $\varepsilon(\nu) = E_C(\Delta N)_0^2 - 2E_J \cos \phi(0)$. Supposing that $\phi(0) = 0$, we obtain the following relation between ν and $(\Delta N)_0$

$$\varepsilon(\nu) = -2E_J + E_C(\Delta N)_0^2. \quad (5)$$

Using Eqs.(4), (5), we arrive at the expressions

$$\nu = \frac{1}{2\pi} |(\Delta N)_0| \quad \text{for the self-trapping (sf) states,}$$

$$\nu = \left(\frac{E_C}{\omega_m} \right) (\Delta N)_0^2 \quad \text{for the "libration" (l) states.} \quad (6)$$

2. The interaction of Josephson states and condensate oscillations can be realized by the following mechanisms. First, the interaction is secured if we allow for the dependence of E_C in $\varepsilon(x)$ on the atom displacements. The latter are generated by the condensate oscillation. Second, the interaction can be realized by applying two-photon traveling-wave laser pulse with the Rabi frequency Ω . The pulse both creates the condensates with

the different particle numbers and induces the interaction of atom's displacements with the excited Josephson states corresponding to the particle number disbalance created by the pulse. General description proposed in what follows is independent on specific mechanism.

Let us consider the classical states of motion of the condensate. These states may be described in terms of the complex amplitudes $a^*, a = n^{1/2} e^{\pm i\varphi_1}$, where $n = \langle a | \hat{a}^\dagger \hat{a} | a \rangle = |a|^2$ is the average number of quanta in the coherent state $|a\rangle$. The variables n, φ_1 are canonical. By classical state of motion we mean that it's number of quanta is very large, $n \gg 1$. It is convenient to specify the relation between the a, a^* amplitudes and the \hat{a}, \hat{a}^\dagger operators by the following way: $a = N^{-1/2} \hat{a}$. At this, the commutator of a, a^* is equal to zero with macroscopic accuracy: $[a, a^*] = 1/N \rightarrow 0$. The Hamiltonian of the motion states can be written in the form: $N\varepsilon(n)$. For the quasiclassical Josephson states with $\nu \gg 1$, the c_ν, c_ν^* amplitudes may be written in the form $c_\nu = \nu^{1/2} e^{i\varphi_2}$. However, it is convenient to rewrite $\varepsilon(\nu), c_\nu$ by means of the variable $x = |(\Delta N)_0|/N^{1/2} \gg 1$. Combining this inequality with the requirement that $x = |(\Delta N)_0|/N \ll 1$ we arrive at the conditions for the x values: $1 \ll x \ll N^{1/2}$. Using Eqs.(4), (5), we find that $\nu = \nu(x)$, $\varepsilon(\nu) = N(-E_J/N + E_C x^2)$. In general case, the $\nu(x)$ dependence is implicit. It is determined by Eqs.(2), (3) и (5). However, in the particular cases of the "libration" ($\varepsilon(\nu) \ll E_J$) and self-trapping ($\varepsilon(\nu) \gg E_J$) states, the relations between ν and $(\Delta N)_0$ can be represented in a simple form, as is seen from (6). By means of Eqs.(5), (6), we come to the following expressions $H_0 \equiv N\varepsilon_0(n, x) = N[-(E_J/N) + \varepsilon(n) + E_C x^2]$ and

$$c_\nu^{(sf)} = N^{1/4} x^{1/2} e^{i\varphi_2} \equiv N^{1/4} c_x^{(sf)}; \quad (7)$$

$$c_\nu^{(l)} = N \left(\frac{E_C}{\omega_m} \right)^{1/2} x e^{i\varphi_2} \equiv N \left(\frac{E_C}{\omega_m} \right)^{1/2} c_x^{(l)}.$$

For any mechanism producing the interaction between two subsystems, it can be written in the form of a multiple Fourier series in φ_1, φ_2

$$\begin{aligned} H_{\text{int}} &= N \sum_{k_1 k_2} [g_{k_1 k_2}(N) a^{*k_1} c_\nu^{k_2} + k.c.] = \\ &= N \sum_{k_1, k_2 > 0} [g_{k_1 k_2}^{(sf, l)}(N) n^{k_1/2} x^{\alpha k_2} e^{i(k_1 \varphi_1 - k_2 \varphi_2)} + k.c.], \\ &\alpha_{sf} = \frac{1}{2}, \quad \alpha_l = 1. \end{aligned} \quad (8)$$

Here k_1, k_2 are integer. For the sake of simplicity, we disregard the phase-independent interaction in Eq.(8). Using Eq.(7), we obtain

$$g_{k_1 k_2}^{(sf)}(N) = gN^{-1+k_2/4},$$

$$g_{k_1 k_2}^{(l)}(N) = gN^{-1+k_2} \cdot \left(\frac{E_C}{\omega_m}\right)^{k_2/2}. \quad (9)$$

The constant g is specified by the concrete mechanism producing the interaction. Let us assume that the term with the phase $\phi_k^r = k_{1r}\varphi_1 - k_{2r}\varphi_2$, which varies anomalously slowly with time, can be set off in sum (8). It is possible to make under two conditions

$$k_{1r} \left(\frac{d\epsilon(n)}{dn}\right) = k_{2r} \left(\frac{d\epsilon_J(x)}{dx}\right),$$

$$\left(\frac{d^2 H_0}{dx^2}\right)_{x=x_m} (\Delta x)_{\max} \gg \left(\frac{\partial H_{\text{int}}}{\partial x}\right)_{x=x_m}, \quad (10)$$

where the x_m value is determined from the first equality in (10)

$$x_m = \frac{k_{1r}\epsilon'_n}{2k_{2r}E_C}, \quad \epsilon'_n \equiv \left(\frac{d\epsilon(n)}{dn}\right). \quad (11)$$

The inequality in (10) is written with taking into account that H_0, H_{int} are functions of a single dynamic variable, e.g., x . The ϵ'_n quantity defines the level separations of an oscillation spectrum. From Eqs.(10), (11) one can obtain that time changing the ϕ_k^r phase is proportional to $(d^2 H_0/dx^2)_m (\Delta x)$, where (Δx) is the change in x near the x_m value. The maximum $(\Delta x)_{\max}$ value specifies the width of the near-minimum region in that $d\phi_k^r/dt \sim \Delta x$. The second condition in (10) implies that the width of the near-minimum region is large at the characteristic interaction variation scale. The estimation for $(\Delta x)_{\max}$ is given in what follows. Thus, the leading term in sum (8) has the form

$$H_{\text{int}}^{(r)} = N g_k^{(sf,l)}(N; n, x) \cos \phi_k, \quad \phi_k = k_{1r}\theta_1 - k_{2r}\theta_2; \quad (12)$$

All remaining terms in this sum are rapidly oscillating perturbations and will be disregarded in this work. Here and below the index k in g_k and ϕ_k denotes a set of k_{1r}, k_{2r} . One can easy to show that aside from energy $H = N[\varepsilon_0(n, x) + g_k(N; n, x) \cos \phi_k]$ the system in question has an additional integral of motion: $n_0 = n/k_{1r} + x/k_{2r}$, $dn_0/dt = 0$. For this reason, the first condition in (10) is equivalent to the condition of the minimum of $\varepsilon_0(n_0, x)$ over x at a given n_0 value. Supposing for the sake of simplicity that $k_{1r} = k_{2r} = k$, one can readily show that the second condition in (10) is

met provided $x_m, (n_0 - x_m) \gg k$. And, for this reason, it is fulfilled with macroscopic accuracy. Using Eqs.(10)-(12), it is straightforward to write the Hamiltonian $H_m = H_0 + H_{\text{int}}^{(r)}$ near the minimum to the first nonvanishing order in (Δx) :

$$H_m = N \left[\varepsilon_0(n_m; x_m) + \left(\frac{d^2 \varepsilon_0}{dx^2}\right)_m (\Delta x)^2 - g_{km} \cos \phi_k \right], \quad (13)$$

$n_m = n_0 - k_{1r}x_m/k_{2r}$, $(d^2 \varepsilon_0/dx^2)_m = 2E_C$, $g_{km} = g_k^{(sf,l)}(N; n_m, x_m)$. Terms with the derivatives of H_{int} are absent in Eq.(13) due to the inequality in (10). Eq.(13) implies that $(\Delta x)_{\max} \sim (g_{km}/E_C)^{1/2}$. Using that $E_C \sim \omega_0(a/a_0)^{2/5}N^{-3/5}$ in the Thomas-Fermi approximation [12] (here $a, a_0 \sim (1/M\omega_0)^{1/2}$ are the scattering and oscillator lengths), we can represent the $1 \ll x_m \ll N^{1/2}$ inequalities in the form

$$\frac{1}{N} \left(\frac{Na}{a_0}\right)^{2/5} \ll \frac{k_{1r}}{k_{2r}} \left(\frac{\epsilon'_n}{\omega_0}\right) \ll \frac{1}{N^{1/2}} \left(\frac{Na}{a_0}\right)^{2/5}. \quad (14)$$

As is known [12], the $(Na/a_0) \gg 1$ relation occurs in the Thomas-Fermi approximation. However, $N^{-1/2}(Na/a_0)^{2/5} \ll 1$. Owing to these relations, the condition (14) or (what is the same) (10) specified the region of the dense oscillation spectrum, where $\epsilon'_n \ll \omega_0$. Both here and in what follows, we suppose that $k_{1r} = k_{2r} = 1$ for the sake of simplicity.

3. At a fixed n_0 value the principal contribution to the partition function comes from the neighbourhood of the minimum at $x = x_m$. The expression for $Z(n_0; x_m; T)$ is equal to

$$Z(n_0; x_m; T) =$$

$$= \text{const} \cdot \int_{-\infty}^{\infty} d(\Delta x) \int_{-\pi}^{\pi} d\phi_k e^{-\beta H_m(n_0, (\Delta x), \phi_k)} \approx$$

$$\approx \frac{\exp[-\beta N \varepsilon_0(N; n_m; x_m) + \ln I_0(\beta N g_{km})]}{(\beta N E_C)^{1/2}}, \quad (15)$$

where $\beta = 1/T$, T is temperature, $I_0(x)$ is the modified Bessel function. Using Eq.(15), we come to the following equation for the \bar{n}_m value realizing the minimum of the free energy

$$\left(\frac{d\varepsilon_0}{dn}\right)_{n_m=\bar{n}_m} = \left(\frac{dg_{km}}{dn_m}\right)_{n_m=\bar{n}_m} \cdot \frac{I_1(\beta N g_{km})}{I_0(\beta N g_{km})},$$

$$I_1(x) = I_0'(x). \quad (16)$$

In addition to \bar{n}_m , the thermodynamic average of $\cos \phi_k$ is determined from Eq.(15). This average is equal to

$$\langle \cos \phi_k \rangle_T = -\frac{\partial \ln Z}{\partial (\beta N g_{km})} = \frac{\partial F}{T \partial (\beta N g_{km})} = \frac{I_1(\beta N g_{km})}{I_0(\beta N g_{km})}. \quad (17)$$

The order parameters \bar{n}_m , $\langle \cos \phi_k \rangle_T$ describe new coherent state. There is the bound state of the \bar{n}_m oscillation quanta and the Josephson state generated by the initial disbalance of the particle number that correspond to the x_m value. In addition, this state has the equilibrium phase coherence factor $\langle \cos \phi_k \rangle_T$. The $\bar{n}_m \neq 0$ value provides the equilibrium distortion of the condensate shape. The aboveobtained equations imply that the shape distortion is self-consistently coupled to the x_m value defining the equilibrium initial disbalance of the particle number.

At $T = 0$ the \bar{n}_m value realizes the minimum of the thermodynamic energy

$$E_m = N[\varepsilon_0(N; n_m, x_m) - g_{km}(N; n_m, x_m)]. \quad (18)$$

To determine the $\bar{n}_m(T = 0)$ value, it is suitable to use the following consideration. As is well known, the level separations ϵ'_n change slowly in dependence on n in the dense (quasiclassical) spectrum. For this reason, we can suppose that $\epsilon'_n \approx \text{const} \equiv \omega_b \ll \omega_0$. On this assumption, the x_m value does not depend on n_m and the \bar{n}_m quantity realizing the minimum of E_m is equal to

$$\bar{n}_m^{1/2} = \frac{g^{(sf,l)}(N)x_m^\alpha}{2\omega_b}, \quad \alpha_{sf,l} = \frac{1}{2}, 1 \quad (19)$$

(it is worth noting that $(\partial^2 E_m / \partial n_m^2) > 0$). Both here and in what follows, the $g_{k=1}^{(sf,l)}(N) \equiv g^{(sf,l)}(N)$ denotation is used. Taking Eq.(19) into account, one readily gets

$$\bar{g}_{1m}^{(sf)} = \frac{g^{(sf)2}(N)x_m}{2\omega_b}, \quad \bar{g}_{1m}^{(l)} = \frac{g^{(l)2}(N)x_m^2}{2\omega_b}, \quad (20)$$

$$\frac{E_m^{(sf)}}{N} = -\frac{E_J}{N} - \frac{g^{(sf)2}(N)}{8E_C} \left(1 - \frac{8E_C^2 x_m^2}{g^{(sf)2}(N)}\right), \quad (21)$$

$$\frac{E_m^{(l)}}{N} = -\frac{E_J}{N} - \frac{g^{(l)2}(N)x_m}{16E_C} \left(1 - \frac{16E_C^2 x_m}{g^{(l)2}(N)}\right).$$

The expressions for energies imply, first, that we obtain the minimum in the region of the dense enough oscillation spectrum, which satisfies the condition (14). The minimum corresponds to the formation of the bound state for the \bar{n}_m, x_m values. Second, as is seen from

Eq.(21), the absolute minimum of E_m can be realized under conditions

$$1 \ll x_m < \frac{g^{(sf)}(N)}{8E_C}; \quad 1 \ll x_m < \frac{g^{(l)2}(N)}{16E_C^2}. \quad (22)$$

These inequalities are met when the interaction matrix elements $g^{(sf,l)}(N)$ are large enough. Let us estimate the condensate parameters that are required for existence of the absolute minimum. In the Thomas-Fermi approximation, the $g^{(sf)}(N) \gg E_C$ inequality occurs provided the total particle number is not very large, namely, $N \ll (g/\omega_0)(a_0/a)^{8/3}$. In turn, the $g^{(l)2}(N) \gg E_C^2$ relation holds true under condition $N^{0.1}[g^2/(\Omega^{1/2}\omega_0^{3/2})](a_0/a)^{0.6} \gg 1$. Here we use that $E_J = \Omega N$. The latter condition is fulfilled for all admissible parameters, provided $g^2/(\Omega^{1/2}\omega_0^{3/2}) \sim 1$. It should be emphasized that the right-side inequalities in Eq.(22) are much stronger than the condition $x_m \ll N^{1/2}$. These conditions imply that the libration Josephson state forms the bound state with the condensate oscillation rather than the self-trapping state.

4. The Hamiltonian (13) describes dynamics of the system that has the ground state with the parameters x_m, \bar{n}_m and phase $\phi_k(0) = 0$ at $t = 0$. In the ground state the initial disbalance of the particle number is equal to $|(\Delta N)_{0m}| = N^{1/2}x_m = N^{1/2}\omega_b/2E_C$. In the case considered here the canonical equations of motion have the form $\partial \phi_k / \partial t = \partial h_m / \partial (\Delta x)$, $\partial (\Delta x) / \partial t = -\partial h_m / \partial \phi_k$, where $h_m = \bar{H}_m / N$. The Hamiltonian \bar{H}_m is obtained from H_m in Eq.(13) at $n_m = \bar{n}_m$ and $g_{1m} = \bar{g}_{1m}$. The \bar{n}_m, \bar{g}_{1m} quantities are determined in Eqs.(19), (20), respectively. The solutions of the equations of motion describe the self-consistent oscillations of the initial disbalance and condensate shape. The first oscillates around the $(\Delta N)_{0m}$ value. The shape of the condensate changes around that which corresponds to the \bar{n}_m value. Note that $(\Delta x) = -\Delta n$. For the libration state the maximum frequency of the oscillations is determined by the following expression

$$\omega_m^{(l)2} = E_C \bar{g}_{1m}^{(l)} \sim g^2 \frac{\omega_b}{\omega_m}, \quad (23)$$

where $\omega_m = (E_C E_J)^{1/2}$ is the maximum oscillation frequency for the "unperturbed" Josephson Hamiltonian (1). In the Thomas-Fermi approximation one readily finds that $E_C \ll \omega_m \ll E_J$. Taking into account that $\omega_b \gg E_C$ (see Eq.(11)) and supposing that $\omega_b \sim E_J$, we obtain

$$\frac{\omega_m^{(l)}}{\omega_m} \sim \left(\frac{\bar{g}_{1m}^{(l)}}{E_J}\right)^{1/2} \sim \left(\frac{g^{(l)}(N)}{E_C}\right) \left(\frac{\omega_b}{E_J}\right)^{1/2} \gg 1. \quad (24)$$

The quite different situation occurs for the self-trapping states. In this case we have

$$\begin{aligned}\omega_m^{(sf)} &= (E_C g_{1m}^{(sf)})^{1/2} \sim g^{(sf)}(N); \\ \frac{\omega_m^{(sf)}}{\omega_m} &\sim \left(\frac{g^{(sf)}(N)}{E_C}\right)^{1/2} \left(\frac{g^{(sf)}(N)}{E_J}\right)^{1/2} \sim \\ &\sim \left(\frac{g^{(sf)}(N)}{E_C}\right)^{1/2} \left(\frac{g}{\Omega}\right)^{1/2} \cdot N^{-7/8}.\end{aligned}\quad (25)$$

With taking the conditions (22) into account, Eq.(25) implies that the ratio $\omega_m^{(sf)}/\omega_m$ may take on an arbitrary value.

It should be emphasized that the bound state arises in the neighbourhood of the minimum of the $\varepsilon_0(n_0; x)$ function, where the conditions (10) are met. The conditions (10) specify the type of the interaction between the Josephson and oscillation degrees of freedom. In addition, the first condition in (10) imposes the definite restrictions on the spectrum of the oscillation states which may effectively interact with the Josephson degrees of freedom. Namely, the density of oscillation states should satisfy the inequalities (14). The region of the dense spectrum exists in the neighbourhood of barrier top. In this case, according to the consideration proposed above, there is formed the equilibrium bound state of the highly excited oscillation and Josephson states. The existence of the bound state generates the equilibrium distortion

of the condensate shape specified by the \bar{n}_m value. This mechanism can provide the experimental detection of the excited Josephson states. In addition, the Eqs.(21) imply that there can be observed the quantum phase transitions to the state with the $\bar{n}_m \neq 0$ value.

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