Vacuum configurations in superstrings associated with semisimple Lie algebras

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A class of vacuum configurations is constructed in superstring theory with a compactification from ten dimensions to four. These configurations are found through a factorization of tori of semisimple Lie groups on the basis of finite symmetry groups. A complete list is given for symmetries generated by a Coxter transformation. The number of generations is calculated.

1. One of the most important aspects of superstring theory[1] is the compactification from ten dimensions to four. It is generally believed that the internal six-dimensional space should be a Kähler and Ricci-planar space,[2] i.e., a so-called Kalabi-Yau space. A Kalabi-Yau space substantially determines the form of the low-energy Lagrangian. The number of generations is thus equal to half the modulus of the Euler characteristic; the result of the breaking of the gauge symmetry depends on the form of the fundamental group[3]; and the Yukawa constants are found from the indices of the intersection of the four-dimensional surfaces in the Kalabi-Yau space.[4] Some examples of Kalabi-Yau spaces have been constructed, although their general classification is not known. Further progress in super string theory depends to a large extent on our understanding of the structure of a Kalabi-Yau space, i.e., of vacuum configurations. It is useful to have as long a list as possible of Kalabi-Yau spaces, containing both simple methodological examples and phenomenologically acceptable cases.

In the present letter we construct some new Kalabi-Yau spaces, working by one of the known methods.[9] The method can be summarized as follows: We consider the three-dimensional complex torus $T^3_{\mathbb{C}}$ which is acted upon by the finite subgroup Γ of the SU_3 group. Fixed points of the Γ group generate singularities in the factorspace $K_0 = T^3_{\mathbb{C}}/\Gamma$. The properties of the algebraic geometry make it possible to smooth over these singularities (to resolve the singularities)[5]; we then find a smooth Kalabi-Yau space K. The choice of the subgroup Γ and the choice of the procedure for resolving the singularities are determined by the circumstance that a Kalabi-Yau space must be obtained as a result. This method for constructing a Kalabi-Yau space has the advantage, in particular, that it becomes possible to extract information about the spectrum of the string without resorting to the complex procedure of resolving singularities, by working with a special manifold K_0, which is relatively simple to construct.[8]

2. We will construct a Kalabi-Yau space from real tori T^6, which are acted upon by large symmetry groups. All of these tori are maximal tori of semisimple Lie groups. (Not all complex tori with symmetries from SU_3 can be found in this manner. We will examine the overall situation in a separate paper.) The torus T^6 is determined by the specification of the periods: the six vectors $\alpha_1,...,\alpha_6$ in $\mathbb{R}^6$, which are the root vectors of the corresponding semisimple Lie algebra G (Ref. 6). This torus is naturally acted upon by the Weyl group W of the algebra G: the group which is generated by the reflections $s_j, j = 1,...,6$ in hyperplanes orthogonal to the simple roots α_j. As Γ here we consider a cyclic subgroup in W: $C = \{1,c,c^2,...,c^{h-1}\}$ $(c^h = 1)$, where $c = s_1...s_6$ is the so-called Coxter element, and H is the Coxter number. Fixed points of the group C are the fundamental weight vectors $\omega_1,...,\omega_6$, but it must be kept in mind that some of these vectors may not coincide on a torus. We can also consider a generalized Coxter transformation which incorporates an external automorphism of a Dynkin graph of the system of roots.[7] The Coxter transformation has l eigenvalues

$$\lambda_j = \exp \frac{2\pi i}{h} m_j, \qquad \lambda_j = \lambda_{l-j+1}, \qquad j = 1, \ldots, l, \tag{1}$$

where l is the rank of the algebra. In our case we have $l = 6$.

3. The construction of the Kalabi-Yau group is broken up into several steps.

a) A description of complex structures on T^6 which are invariant under the group C. This point causes no difficulty. According to (1), the complex coordinate z_k corresponds to the eigenvalue λ_{j_k}, $k = 1, 2, 3$, while the conjugate coordinate $\bar{z}_k$ corresponds to the eigenvalue $\bar{\lambda}_{j_k}$. The choice of three complex coordinates transforms the torus T^6 into the complex torus $T^3_{\mathbb{C}}$.

b) The choice from all possible complex structures of that in which the holomorphic 3-form $dz_1 \wedge dz_2 \wedge dz_3$ on $T^3_{\mathbb{C}}$ is invariant under C. In this case, there will be a holomorphic 3-form on the special factorspace $K_0 = T^3_{\mathbb{C}}/C$. Many algebras are accordingly ruled out.

c) Resolution of the singularities. Instead of the singular points corresponding to fixed points of the powers of the Coxter element c on $T^3_{\mathbb{C}}$, we paste in some complex spaces consisting of, in general, several complex surfaces that intersect in some way. For cyclic three-dimensional factor singularities which meet the requirement of the preceeding step (the existence of holomorphic 3-forms) there always exists a resolution such that there is also a nonzero 3-form on it. Resolving the singular points K_0 by this canonical method, we find the Kalabi-Yau space K.

d) Calculation of the topological characteristics of K. The most important of these characteristics is the Euler characteristic, whose modulus, divided by two, is equal to the difference between the number of generations and antigenerations.[2,8] Dixon *et al.*[8] have suggested that the Euler characteristic $\chi(K)$ can be regarded as simplified by considering a string on a special manifold K_0. Denoting by h the order of the group Γ, and denoting by $\chi(g_1, g_2)$ the Euler characteristic of the manifold consisting of the fixed points of the commuting transformations g_1 and g_2, we can write

$$\chi(K) = \frac{1}{h} \sum_{g_1 g_2 = g_2 g_1} \chi(g_1, g_2). \tag{2}$$

This formula can be proved rigorously by the method of toric geometry in the case in which the group Γ is Abelian, in accordance with the situation under consideration here.

TABLE I .

Symmetry	Indices	$N = \|\pi_1(K)\|$	$\chi(K)$
1. $A_2^{(1)} \times A_2^{(1)} \times A_2^{(1)}$	$\frac{1}{3}(111)$	27	72
2. $A_2^{(1)} \times D_4^{(1)}, A_1^{(1)} \times A_5^{(1)}$	$\frac{1}{6}(123)$	12	48
3. $B_2^{(1)} \times B_4^{(1)}, B_2^{(1)} \times D_4^{(2)}$	$\frac{1}{8}(125)$	4	48
4. $B_4^{(1)} \times D_2^{(1)}, D_2^{(1)} \times D_4^{(2)}$	$\frac{1}{8}(134)$	8	48
5. $A_2^{(1)} \times D_4^{(3)}, A_2^{(1)} \times F_4^{(1)}, E_6^{(1)}$	$\frac{1}{12}(147)$	3	48
6. $A_2^{(1)} \times G_2^{(1)} \times G_2^{(1)}$	$\frac{1}{6}(114)$	3	48
7. $D_2^{(1)} \times F_4^{(1)}$	$\frac{1}{12}(156)$	4	48
8. $A_6^{(1)}$	$\frac{1}{7}(124)$	7	48
9. $A_3^{(1)} \times A_3^{(1)}$	$\frac{1}{4}(112)$	16	48

The Yukawa constants must be calculated on the complete manifold K. Analysis of examples show that 85–95% of the Yukawa constants vanish.

Another important characteristic is the fundamental group $\pi_1(K)$. This group tells us the number of noncontractible contours in K. Nonzero values of Wilson integrals over such contours have the same effect as Higgs fields in the associated representation of the gauge group.[2] The structure of the group $\pi_1(K)$ thus tells us about the nature of the breaking of the gauge symmetry. It can be proved that in cases with $\chi(K) \neq 0$ the order of the fundamental group $\pi_1(K)$ is equal to the value of the characteristic polynomial $P(x)$ of element c at point $x=1$, i.e., the number (N) of fixed points of the element c on[3)] T^6. The fundamental group is calculated by the Van Kampen method.

4. Let us list the semisimple algebras from which Kalabi-Yau spaces are found by this procedure. The results are shown in Table I. The Kalabi-Yau spaces on the first line were discussed in Refs. 2 and 3. In the first column of this table, the superscript in the specification of the Dynkin graph means the order of its external automorphism. The second column contains information on the action of the element c on $T^3_{\mathbb{C}}$ in the form $1/h(a_1, a_2, a_3)$, where h is the order of c (i.e., the Coxter number), while the indices a_j prescribe the choice of the complex coordinates z_1, z_2, z_3 on T^6 at which the action of c takes the form $(z_1, z_2, z_3) \to (\epsilon^{a_1} z_1, \epsilon^{a_2} z_2, \epsilon^{a_3} z_3)$, $\epsilon = e^{2\pi i/h}$.

5. In addition to the subgroups generated by the Coxter element, we could consider other subgroups of W. For example, a factorization of the algebra E_6 on the basis of the cyclic group $\mathbb{Z}_3$, which permutes the branches of the expanded Dynkin diagram, leads to a Kalabi-Yau space K with $\chi(K) = 0$, which stratifies with a surface layer of the K3 type on a one dimensional complex torus.

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[3)]In this case, the gauge group E_6 is broken with conservation of rank,[3] e.g., to $Su_3^c \times SU_2 \times U_1 \times U_1 \times U_1$.

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