Equivalence of the Ising model and the fermion model on a Riemann surface

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The Ising model on an irregular triangulated lattice of arbitrary topology is represented in terms of Majorana fermions which "live" on a Riemann surface.

The possibility of modeling fermion degrees of freedom of a string by means of Ising spins has attracted interest. This possibility is based on the circumstance that both a system of Ising spins and fermions which "live" on a surface and which interact with its curvature give rise, in the expression for the partition function, to a summation over closed, non-self-intersecting paths on the surface in addition to a summation over random surfaces.[1] This topic is currently of interest in connection with research in the theory of random triangulated surfaces[2] whose metric properties are determined by the type of triangulation (e.g., the scalar curvature at a given point i is $(\pi/n_i)(n_i-\sigma)$, where n_i is the number of links which come from point i).

In this letter we report the construction of a fermion representation for the Ising model on an irregular triangulated lattice of arbitrary topology. Let us examine the basic results.

1. It has been proved that the Ising model with spins $\sigma = \pm 1$ at the vertices of triangulated lattice Γ, with the Hamiltonian

$$H=-h\sum_{\langle ij\rangle}\sigma_i\sigma_j \tag{1}$$

($\langle ij\rangle\in\Gamma$ are links of the lattice), is equivalent to the system of Majorana fermions ψ_α ($\alpha = 1,2$) at the vertices of the dual graph Γ^* (Fig. 1), which is described by the action

$$S(\psi)=\sum_{\langle ij\rangle\in\Gamma^*}\bar\psi_\alpha(i)P_{\alpha\beta}(ij)\psi_\beta(j)+\sum_{k\in\Gamma^*}m\bar\psi_\alpha(k)\psi_\alpha(k), \tag{2}$$

$$m=e^{-2h}, \tag{3}$$

$$\bar\psi_\alpha=\epsilon_{\alpha\beta}\psi_\beta\,. \tag{4}$$

Any lattice with the topology of a sphere can be projected without self-intersections onto a plane (a Riemann surface of type 0). Each link acquires a two-dimensional direction $\pm\mathbf{n}(ij)$ (the choice of sign is not important). We introduce the two-dimensional matrix vector $[\vec\sigma=(\sigma_1,\sigma_3)]$ (σ_i are the Pauli matrices). We then have

$$P_{\alpha\beta}(ij)=\alpha(ij)\,\{1+(\mathbf{n}(ij)\,\vec\sigma)\}\,. \tag{5}$$

The coefficients $\sigma(ij)$ depend on the four angles adjacent to link $\langle ij\rangle$ in the plane; they are analogs of a spin-connection (Fig. 1). The matrices $P_{\sigma\beta}(ij)$ have two important properties:

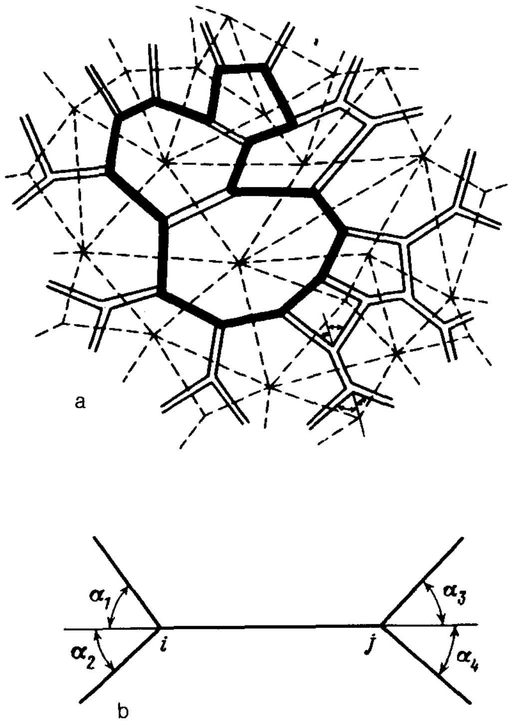

FIG. 1 (a)—The dashed line shows a lattice Γ with spins σ at its sites, the double-dual graph Γ^* with fermions at the vertices. The fermion contour $\gamma(i_1 ... i_n i_1)$ bounds the region of constant spin and denotes one of the terms in the expansion of Z_ψ in loops (the heavy line). Associated with the links γ are corresponding matrices P; (b)—the factor $\alpha(ij)$ on the link $\langle ij\rangle \in \Gamma^*$ depends on the angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

A. For any closed contour on the dual graph, $\gamma(i_1 i_2 ... i_n i_1)$, we can write

$$\mathrm{Sp} P(i_1 i_2) P(i_2 i_3) \ldots P(i_n i_1) = -1. \quad (6)$$

B. For any link $\langle ij\rangle \in \Gamma^*$ we can write

$$P(ij) P(ij) = 0. \quad (7)$$

When these properties hold, we have

$$Z_\sigma(h) = \sum_{\{\sigma\}} e^{H(h,\sigma)} = Z_\psi^{(0)} = \int \prod_{i,\alpha} d\psi_\alpha(i) e^{-S(\psi)} \quad (8)$$

The second property of the matrices P means that the sum over the nonintersecting contours, in the form of which the Berezin integral in (8) is represented, does not contain any contours that do not contain vertices of the lattice Γ.

In the case of a correct triangulated lattice modeling a surface without an internal curvature, the action $S(\psi)$ is the same as that derived previously by Dotsenko and Dotsenko.[3]

2. The result of Section 1 has been generalized to the case of a surface laboratory topology.

A fermion path must bound a region of constant spin and must therefore be homologous to zero. As a result, the partition function $Z_\psi^{(g)}$ for a surface of type g differs in form from $Z_\psi^{(0)}$ because of the nontrivial nature of the group of homologies H_1^g. To construct $Z_\psi^{(g)}$, we need to project the surface onto g copies of the parallelo-

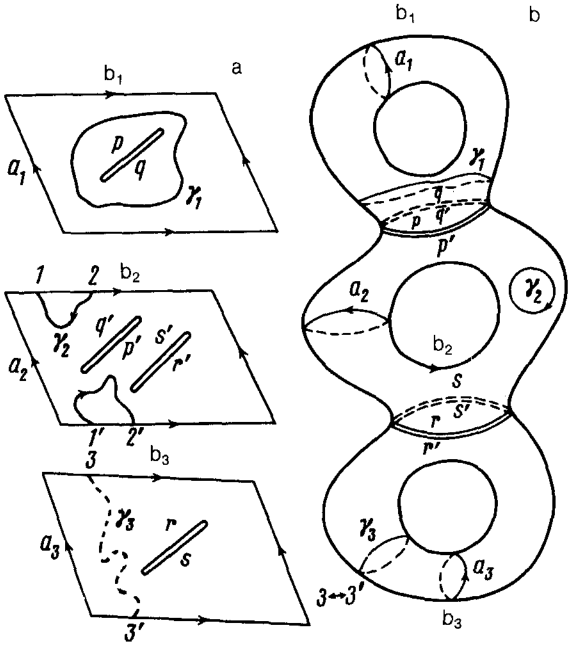

FIG. 2. (a)—The three sheets of the Riemann surface represent a surface with three handles. The sides of the cuts ($p\leftrightarrow p', q\leftrightarrow q', \ldots$) and the opposite sides of the parallelograms are associated with each other during the splicing. The solid line shows loops which are homologous to zero and which enter partition function $Z_\psi^{(3)}$. The loop shown by the dashed line appears in $Z_\psi^{(0)}$ with a factor $\epsilon_{b_3} = \pm 1$ and drops out of $Z_\psi^{(3)}$; (b)—a surface of type $g = 3$.

gram $(\mathbf{a}_i, \mathbf{b}_i)$ $(i = 1, \ldots g)$, i.e., onto the g sheets of the Riemann surface, factorized on the basis of a discrete group of translations specified by the vectors $\{\mathbf{a}_i, \mathbf{b}_i\}$. Cuts are made on the sheets of the Riemann surface (the sides marked in the figure are associated with each other when the splicing is carried out; Fig. 2).

A fermion field is not a single-valued function on a surface of type g. Any fermion determinant (including $Z_\psi^{(0)}$ on this surface depends on $2g$ boundary conditions specified at the sides of the parallelograms. The boundary conditions are specified by a set of sign factors $\{\epsilon_{a_i}, \epsilon_{b_i}\}$, where $\epsilon_{a_i} = \pm 1$, $\epsilon_{b_i} = \pm 1$. If x belongs to a side of a parallelogram, then we have

$$\psi(x) = \epsilon_{ai}\, \psi(x + a_i) \quad \text{or} \tag{9}$$

$$\psi(x) = \epsilon_{b_i}\, \psi(x + b_i). \tag{10}$$

The correct partition function is a superposition of the determinants for all possible $\{\epsilon_{a_i}, \epsilon_{b_i}\}$:

$$Z_\psi^{(g)} = \frac{1}{4^g} \sum_{\{\epsilon_{a_i}, \epsilon_{b_i}\}} Z_\psi^{(0)} (\epsilon_{a_i}, \epsilon_{b_i}). \tag{11}$$

This expression was cited by Regge *et al.*[4] for the particular case of a regular lattice with the topology of a torus $(g = 1)$.

The following assertion holds: The partition function $Z_\psi^{(g)}$ in (11) can be represented as a sum over closed (not necessarily connected) non-self-intersecting paths on the graph Γ^* which are homologous to zero. If a term of the sum contains paths on more than one Riemann sheet, then there is an even number of such sheets. (The latter condition is required for satisfaction of the first property of the matrices P.)

When these two conditions on the matrices $P(ij)$ are satisfied, the partition function of the fermion system, $Z_\psi^{(g)}$, is $e^{hN} \Sigma_{L=0}^{\infty} e^{-2hL} M(L)$ [N is the number of points of the lattice, $M(L)$ is the number of non-self-intersecting contours of length L on the dual graph Γ^* which are closed and homologous to zero] and is exactly the same as the partition function of a system of Ising spins, $Z(\sigma)$.

Remarkably, $Z_\psi^{(g)}$ does not depend on nonphysical parameters of the fermion model ($\mathbf{a}_i$, $\mathbf{b}_i$, α_{ij}, $\mathbf{n}_{ij}$, etc.). It is determined exclusively by the topology of the lattice and by the type of triangulation (the numbers n_i, for example).

In contrast with the square lattice examined by Sherman and Vdovichenko,[5] a triangulated lattice automatically ensures that there are no self-intersections of paths if the action is quadratic on fermion fields. Of the three links coming from a vertex $i \in \Gamma^*$, either two or zero are fermion lines (Fig. 1). A detailed derivation of the basic results will be published separately.

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