Theta-function representation of the partition function of a Polyakov string

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An expression is given for a Polyakov measure on a space of Riemann surfaces of the type $p \geqslant 2$ in terms of theta functions.

The quantum theory of a Polyakov string[1] is determined by a path integral over surfaces in a d-dimensional Euclidean space. The partition function for a closed string is written as a "perturbation-theory" series $Z = \Sigma_{p \geqslant 0} Z_p$. The term Z_p corresponds to a p-loop contribution and is of the form $\int_{M_p} d\pi_p$, where M_p is the space of conformal classes of compact Riemann surfaces of type p (a sphere with p handles), and $d\pi_p$ is some measure on it.[2,3] Our purpose in the present letter is to give expressions for $d\pi_p$ in the critical dimensionality $d = 26$ (and the analogous measure, $d\pi_p^f$, for a fermion string with $d = 10$) in terms of theta functions.[6] Expressions which have recently been proposed for $d\pi_p$ and $d\pi_p^f$ have been found through a transformation of the determinants of two elliptical operators into a sum over the length of geodesics by means of the Selberg trace formula.[3] Our approach in the present paper is entirely different. It is based on a holomorphic, rather than Riemannian, geometry. This approach became possible after the well-known study by A. Belavin and V. Knizhnik, who established that $d\pi_p$ is the "square modulus" of a holomorphic form on M_p with known singularities. The formulas given in Ref. 3 and the present letter express the same quantity in terms of different invariants of the Riemann surface. Both may be useful for finding the properties of a Polyakov measure. Our calculations lean heavily on papers by Mumford[4] and Faltings.[5]

Our basic expression, for $p \geqslant 2$, is as follows (the case $p = 1$ is much simpler and has been resolved previously):

$$d\pi_p = c_p K (-i)^{3p-3} W_1 \overline{W}_1 \ldots W_{3p-3} \overline{W}_{3p-3}, \tag{1}$$

$$K = (\det \operatorname{Im} \tau)^{-4} |\theta(0,\tau)|^{-16} \frac{|\det(\omega_j(R_k))|^{16} \prod_{j=1}^{p} G(R_j, Q)^{16} \prod_{j<k} G(P_j, P_k)^4}{|\det(\beta_i(P_j))|^2 \left|\det\begin{pmatrix} \eta_i(P_j) \\ a_j \end{pmatrix}\right| \prod_{j<k} G(R_j, R_k)^{16}} \cdot$$

Here the W_i are (1,0)-forms on the space M_p which correspond to some basis w_i of (quadratic) holomorphic differentials on Riemann surface X,τ is the normalized matrix of periods of this surface, θ is its theta function, $\log G$ is the Green's function of the invariant Laplacian, and P_j,R_k, and Q are some special points on X.

Let us write expressions for these entities in slightly more detail (nearly all the details are given in Ref. 6). We assume that X is a Riemann surface which depends on the parameters in a certain region M_p. We write it as a polygon, making a classical system of cuts (a_i, b_i), $i = 1, \cdots, p$. We choose a basis of holomorphic 1-forms $\varphi = (\varphi_i)$ with $\int_{a_i} \varphi_j = \delta_{ij}$, and we set

$$\tau = (\tau_{ij}) = (\int_{b_i} \varphi_j), \omega_i = \sum_j (\sqrt{\operatorname{Im} \tau})^{-1}_{ij} \varphi_j.$$

The Riemann theta function is defined by the series

$$\theta(z, \tau) = \sum_{m \in Z^p} \exp 2\pi i [m^t z + \frac{1}{2} m^t \tau m] .$$

We set $\mu = (i/2p) \Sigma_{j=1}^p \omega_j \wedge \overline{\omega}_j$, and we introduce an invariant Laplacian on X by means of the formula $\Delta f = -(i/\pi\mu) \partial \overline{\partial} f$, where

$$\partial \overline{\partial} f = \frac{1}{4} dz \wedge d\overline{z} (\partial_x^2 + \partial_y^2), \quad \text{and} \quad z = x + iy$$

is the local coordinate on X. We determine $G(P,Q)$ from the expression $f(P) = -\int_X \log G(P,Q) \Delta f(Q) \mu(Q)$. We can also express $G(P,Q)$ in terms of a theta function and Abelian integrals, but we will not reproduce this expression here. On X we choose points $P_1, \cdots, P_{p-1}$ (which depend on the parameters) in such a way that a holomorphic 1-form with second-order zeros at all the P_i exists on X (this can be done in a finite number of ways). We denote this form by ν^2. We normalize it by the condition $|\nu^2(P)| = \prod^{p-1} G(P_i, P)$ for all P (more on this below). Using ν, we can introduce differentials of half-integer order on X. We choose the bases of holomorphic differentials on X of order 3/2 in the form $(v_1, \ldots, v_{2p-2}) = (\nu\omega_1, \ldots, \nu\omega_p; \eta_1, \ldots, \eta_{p-2})$ and of order 2 in the form $(w_1, \ldots, w_{p-3}) = (\nu^2 \omega_i; \nu\eta_j; \beta_1, \ldots, \beta_{p-1})$. If $\omega = f(z)dz$, we set

$$|\omega(P)| = |f(P)| \lim_{Q \to P} G(P, Q) \cdot |z(Q) - z(P)|^{-1}.$$

For differentials of order m, we set $|\omega(P)| = |(\omega)^{1/m}(P)|^m$. We determine the local coordinates z_j at the points P_j by the condition $\nu^2 = z_j^2 dz_j$ and we set $a_j = (\nu z_j^{-1}(P_j))^3$. Finally, we choose those points $R_1, \cdots, P_p$, Q on X such that there exists a meromorphic 1-form on X with second-order zeros at $R_1, \cdots, R_p$ and with a second-order pole at Q. In doing so, we require that the class $R_1 + \cdots + R_p - Q$ be a Riemann class Δ (p. 7 in Ref. 6). This completes the description of all the notation in (1) except the constant c_p, which is not determined here. A joint normalization of all the c_p can apparently be carried out, so that an asymptotic consistency of measures is achieved in the limit in which X acquires a double point.

The measure for a fermion string has an analogous structure:

$$d\pi_p^f = c_p^f L (-i)^{5p-5} \prod_{i=1}^{3p-3} W_i \overline{W}_i \prod_{j=1}^{2p-2} \partial^2 / \partial V_i \partial \overline{V}_i, \tag{2}$$

$$L = (\det \operatorname{Im} \tau)^{-5/2} |\theta(0,\tau)|^{-10} \frac{|\det(\omega_j(R_k))|^{10} \prod_{j=1}^{p} G(R_j, Q)^{10} \prod_{j<k} G(P_j, P_k)^2}{|\det(\beta_i(P_k))|^2 \prod_{j<k} G(R_j, R_k)^{10}}$$

This measure is determined on the complex superspace M_p^f, whose substrate coincides with M_p, and the odd local analytic coordinates are the differentials $(v_1, \cdots, v_{2p-2})$ of order 3/2 on X (cf. Ref. 3). We also denote the dual basis of odd vector fields on M_p^f by $(\partial/\partial V_i)$. Consequently, the Berezin volume element in (2) is shown in the notation $dxdy\partial_\xi \partial_\eta$, rather than $dxdyd\xi d\eta$; this notation is more successful in conveying the transformation rule upon a change in coordinates. The other notation in (2) is the same as in (1). Strictly speaking, expression (2) should be regarded as hypothetical until the analog of the Belavin-Knizhnik theorem is proved for a fermion measure. Furthermore, the superspace M_p^f has yet another connected component, on which expression (2) should change slightly. I hope to return to these questions in a future paper.

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[1]A. M. Polyakov, Phys. Lett. **103B**, 207 (1981).
[2]O. Alvarez, Nucl. Phys. **B216**, 125 (1983).
[3]M. A. Baranov and A. S. Shvartz, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 340 (1985) [JETP Lett. **42**, 419 (1985).
[4]D. Mumford, Ens. Math. **23**, 39 (1977).
[5]G. Faltings, Ann. Math. **119**, 387 (1984).
[6]J. D. Fay, Theta Functions on Riemann Surfaces, Springer Lecture Notes in Math., 352, Berlin, 1973.

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