Theory of quasi-Chaplygin unstable media and evolutionary principle for selecting spontaneous solutions

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Many unstable media (about 20) can be described by the ideal gas equations with a *negative compressibility*. These "quasi-Chaplygin" equations can be reduced to a Laplace equation. Three "spontaneous-evolution" solutions are of primary interest.

1. "QUASI-CHAPLYGIN" MEDIA

We consider a one-dimensional ideal gas whose pressure is described by the adiabat $p(t,x) = p_0(\rho/\rho_0)^\gamma$, where $\rho(t,x)$ is the density, and $p_0 = \text{const}$ and $\rho_0 = \text{const}$ are unperturbed values. If we assume $\gamma = -|\gamma|$ and introduce the "effective reduced density" $\bar{\rho} = \rho/\rho_0$, we find equations for the velocity $v(t,x)$ and for $\bar{\rho}(t,x)$:

$$\frac{\partial}{\partial t}\bar{\rho} + \frac{\partial}{\partial x} v\bar{\rho} = 0; \quad \frac{\partial}{\partial t} v + v\frac{\partial}{\partial x} v = c_0^2 (\bar{\rho})^{-\nu} \frac{\partial}{\partial x}\bar{\rho}, \tag{1}$$

where $\nu = 2 + |\gamma|$ and $c_0^2 = |\gamma| p_0/\rho_0$. Below, however, we will treat ν and c_0 as arbitrary real parameters. We have noted that in the long-wave approximation equations of this type describe the behavior of an extremely large number (but, of course, not all) unstable media, in particular, the following:

1) a "Chaplygin gas" with $\gamma = -1$ (first studied in Ref. 1),
2) "tipped" shallow water,"[2–5]
3) constrictions in an incompressible pinch,[4–7]
4) constrictions in a compressible pinch,[4–6]
5) a cylinder of a liquid with surface tension,[4,5]
6) self-focusing of light,[8–10]
7) self-contraction of wave packets,[10]
8) tearing-mode instability of a plasma current sheet,[11]
9) modulation instability in a plasma,[12]
10) Buneman instability in a plasma,[13,14]
11) beam instability in a plasma,[15]
12) aperiodic parametric instability of a plasma,[16]
13) instability of a gravitating gas slab.[17]

There are several other examples. We call those media which can be described by system (1), and which constitute a rather large set, "quasi-Chaplygin" media, since Chaplygin, in 1902, was the first to examine the particular case of a gas with $\gamma = -1$ (Ref. 1).

2. REDUCTION TO A LAPLACE EQUATION

We can show that system (1) reduces, for arbitrary $\nu \neq 1$, to a Laplace equation in a three-dimensional "phase" space. For this purpose, we replace $\bar{\rho}$ and v by the new dimensionless functions

$$r(t,x)=(\bar{\rho})^{-1/2m}>0; \quad z(t,x)=v/2mc_0, \tag{2}$$

where $m=1/(\nu-1)$ is the "azimuthal number." System (1) then gives us

$$\frac{\partial r}{c_0\partial t}=r\frac{\partial z}{\partial x}-2mz\frac{\partial r}{\partial x}; \quad \frac{\partial z}{c_0\partial t}=-r\frac{\partial r}{\partial x}-2mz\frac{\partial z}{\partial x}, \tag{3}$$

and if we seek the inverse functions $t=T(r,z)$ and $x=c_0X(r,z)$ (i.e., if we take a "hodograph transformation"), we find the two linear equations

$$X_r'=2mzT_r'-rT_z'; \quad X_z'=2mzT_z'+rT_r'. \tag{4}$$

The condition for the compatibility of these equations is an equation for the time:

$$T_{rr}''+\frac{1-2m}{r}T_r'+T_{zz}''=0. \tag{5}$$

Supplementing r and z with the "angle" φ, we treat r, φ, z as cylindrical coordinates in a three-dimensional "phase" space, and we introduce an analog of the "electrostatic potential": $\Psi=\psi\cos m\varphi$, where $\psi(r,z)=r^{-m}T$. From (5) we then find a Laplace equation:

$$\Delta\Psi=\frac{\partial}{r\partial r}r\frac{\partial}{\partial r}\Psi+\frac{\partial^2}{r^2\partial\varphi^2}\Psi+\frac{\partial^2}{\partial z^2}\Psi=0. \tag{6}$$

Particular solutions of this equation have actually been found in several of the papers cited above on the various instabilities. For example, polynomial solutions were examined for the case $\nu=0$ in Ref. 9.

3. "EVOLUTIONARY PRINCIPLE" FOR SELECTING SPONTANEOUS SOLUTIONS

Although a study of particular solutions does give us definite information on the behavior of the medium, such solutions generally describe the evolution of an "initial jolt" provided by the researcher. In a numerical solution method, an initial-value problem is the only possible formulation of the problem. In this approach, the profile of the initial perturbation is chosen by the investigator for convenience and simplicity in the solution. In unstable media, however, in contrast with stable media, spontaneous perturbations can develop. In the limit $t\to-\infty$, there are no such perturbations, so that the asymptotic "initial conditions" $v=0$ and $\rho=\rho_0$ hold; these conditions correspond to $r=1$ and $z=0$ in the limit $t\to-\infty$. For these special perturbations, which might be called "spontaneously evolving," the electrostatic potential $\Psi(r,\varphi,z)$ introduced above should be generated by only those "charges" which lie on a unit circle $r=1$, $z=0$ in the r,φ,z phase space. We call this requirement an "evolutionary principle" for the selection of spontaneous solutions.

4. THREE EXTREMELY SIMPLE SPONTANEOUS SOLUTIONS

To satisfy this evolutionary principle, we should solve Laplace equation (6) in toroidal coordinates ζ, η, φ, introduced by

$$r=\sigma \sinh \xi \; ; \quad z=\sigma \sin \eta ; \quad \sigma=(\cosh \xi+\cos \eta)^{-1} ; \quad (dr)^2+(dz)^2=(\sigma d\xi)^2+(\sigma d\eta)^2, \tag{7}$$

and we should seek solutions of the type

$$\Psi(\xi, \eta, \varphi)=\psi(\xi, \eta) \cos m\varphi \; ; \quad \psi=\frac{1}{\sqrt{r}} \sum_{n=0}^{\infty} a_n Q_p^n(\alpha) \cos n\eta, \tag{8}$$

since only solutions of this type are generated by "charges" (circular multipoles) on the circle $r=1$, $z=0$. In Eq. (8), the a_n are arbitrary coefficients of the "second" Legendre functions $Q_p^n(\alpha)$ with an integer superscript $n=0,1,2,....$; a subscript $p=m-1/2$; and an argument $\alpha=\coth \xi>1$. The simplest terms are the first two terms, the Coulomb and dipole terms, which generate the three "fundamental" solutions

$$\begin{aligned} t/t_* &= T_1(\xi, \eta)=-r^p Q_p^0(\alpha)<0 && (-\infty<\eta<\infty) \\ t/t_* &= T_2(\xi, \eta)=r^p Q_p^1(\alpha) \cos \eta<0 && (-\pi/2<\eta<\pi/2) \\ t/t_* &= T_3(\xi, \eta)=-r^p Q_p^1(\alpha) \cos \eta<0 && (\pi/2<\eta<3\pi/2) \end{aligned} \quad , \tag{9}$$

where $t_*>0$ is the sole parameter, which specifies the scale time of the evolution and simultaneously characterizes the length of the perturbation along the x axis. Equations (4) for the coordinate can be put in the form

$$\nabla X=2mz\nabla T-r[e_\varphi \nabla T] \; ; \quad X_\eta'=2mzT_\eta'-rT_\xi' \tag{10}$$

Integrating over the angle η here, we can easily find the coordinate $x=c_0 X(\xi,\eta)$ for the three cases in (9), but we will not go into detail here. The first case, T_1, gives us a perturbation which is periodic in x; the second, T_2 (a "left-hand dipole"), gives us an isolated well in the parameter r; and the third case, T_3 (a "right-handed dipole"), gives us an isolated hill in the parameter r. These three cases give a graphic description of the simplest types of spontaneous evolution of all of the unstable media listed above—whose nonlinear description in the long-wave approximation can be reduced to a "quasi-Chaplygin" system (1).

5. MULTIDIMENSIONAL SELF-SIMILAR SOLUTIONS

In several cases, the problem can be formulated not only as a one-dimensional problem ($N=1$) but also as a two-dimensional ($N=2$) or three-dimensional ($N=3$) problem. With a symmetry for these three cases ($N=1,2,3$), the equations should be written in the form

$$\frac{\partial}{\partial t}\bar{\rho}+r^{1-N}\frac{\partial}{\partial r}(\bar{\rho} v r^{N-1})=0; \quad \frac{\partial}{\partial t}v+v\frac{\partial v}{\partial r}=c_0^2(\rho)^{-\nu}\frac{\partial}{\partial r}\bar{\rho}, \tag{11}$$

and for them we can find exact self-similar solutions, although these solutions do not satisfy the evolutionary principle $\tau=t-t_*<0$, $a=2m/(2m-N)$:

$$v=a\frac{r}{\tau} \; ; \quad \bar{\rho}=\left[(\bar{\rho}_0)^{-1/m}-N\left(\frac{v}{2mc_0}\right)^2\right]^{-m} ; \quad \bar{\rho}_0(t)=\left|\frac{\tau_0}{\tau}\right|^{aN}. \tag{12}$$

These solutions correspond to completely definite initial conditions at $t=0$, and they "explode" ($\bar{\rho}\to\infty$) at the time $t=t_*$.

6. EXAMPLE OF A NONLINEAR BUNEMAN INSTABILITY

As an example we consider the Buneman instability in a plasma (following Ref. 14), which arises when the electrons (e) are initially moving at a suprathermal velocity $v_0 > v_{\rm Te}$ with respect to the ions (i), which are initially at rest and are then described by the equations

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} nv = 0; \quad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{e}{m_i}E = -\frac{e}{m_i}\frac{\partial \varphi}{\partial x}, \tag{13}$$

where $v = v_i$, $n = n_i = n_e$, and φ is the electric potential. For electrons we can assume that the total flux is conserved ($nv_e = n_0 v_0$) (the electron current), as well as the energy, $m_e v_e^2/2 - e\varphi = m_e v_0^2/2$. We can then put system (13) in quasi-Chaplygin form, (1):

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x}\bar{\rho} v = 0; \quad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = c_0^2(\bar{\rho})^{-3}\frac{\partial \bar{\rho}}{\partial x}, \tag{14}$$

where $\bar{\rho} = n/n_0$, $c_0^2 = v_0^2\, m_e/m_i$, and the value of the parameter ν is 3, as for a Chaplygin gas, so that the "azimuthal number" is $m = 1/2$, and we have $p = 0$. The first "Coulomb" solutions of (9) and Eq. (4) then give us

$$\frac{t}{t_*} = T_1 = -Q_0^0(\alpha) = -\xi; \quad \frac{x}{c_0} = X = 2t_* \arctan\left(\tanh\frac{\xi}{2}\tan\frac{\eta}{2}\right). \tag{15}$$

If we eliminate the variables ξ and η, we find expressions, all periodic in x, for the density of the quasineutral plasma, the ion velocity, and the potential of the Buneman instability:

$$\frac{n}{n_0} = \frac{-\sinh\tau}{\cosh\tau - \cos\chi}; \quad \frac{v}{c_0} = \frac{\sin\chi}{\sinh\tau}; \quad \varphi = \frac{1 - 2\cosh\tau\cos\chi + \cos^2\chi}{\sinh^2\tau}\left|\frac{m_e v_0^2}{2e}\right|, \tag{16}$$

where $\tau = t/t_* < 0$, $\chi = x/c_0 t_*$. This extremely simple solution, which is explicit and which satisfies the evolutionary principle, was not mentioned in Ref. 14 or in Ref. 11, where a study was made of the tearing-mode instability of a current sheet in a plasma, which is also analogous to a Chaplygin gas. The last two solutions in (9) describe an isolated well and an isolated hump in the density in both of these problems.

Our three standard evolutionary solutions in (9), which evolve spontaneously, thus provide a useful supplement to the picture of all of the instabilities listed in Section 1, although the particular solutions found previously by other investigators are also of obvious value.

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