Two- and three-loop amplitudes in boson string theory

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Explicit expressions are derived for the two- and three-loop vacuum amplitudes in the theory of oriented closed boson strings with $D = 26$ in terms of the theta constants. The space of moduli is parametrized by means of period matrices.

The problem of calculating multiloop amplitudes in string theory has recently attracted much interest. The basic "practical" goal of this research is to prove that the theory of superstrings is finite, although the possibility is not ruled out that a complete understanding of the structure of the multiloop corrections would also make it possible to make progress in the solution of other problems. In any event, the model of oriented closed boson strings (the ESVM) is a good laboratory for work in this direction.

In the present letter we derive explicit expressions for the two- and three-loop vacuum amplitudes in the ESVM in the critical dimensionality $D = 26$. As has been shown elsewhere,[1] this problem reduces to the search for the measure on the space M_p of complex structures of Riemann surfaces of type p (in the present letter, $p = 2,3$). We know that for $p \geqslant 2$ this space has a dimensionality $6p - 6$ and is a complex manifold. Analytic properties of this measure were found in Ref. 2 as functions of the complex coordinates $y_1, \ldots, y_{3p-3}$ on M_p, and it was shown that the properties which have been found determine the measure unambiguously with an accuracy to within a constant factor. The basic difficulty in describing the explicit expression for the measure in the case of arbitrary p is that we do not have a good parametrization of the space M_p. In the cases $p=2, 3$ (and, of course, $p=1$), however, the complex structures can be parametrized by period matrices. This circumstance and the analytic properties of the measure which were found in Ref. 2 make it possible to express this measure in terms of theta constants.

1. Analytic properties of the measure.[2] The complex coordinates y_i in the space M_p are introduced in the following way.[3] We consider a Riemann surface S_p of type $p\geqslant 2$ with the coordinates ξ^1, ξ^2 and the metric $ds^2=g_{ab}d\xi^a d\xi^b$. Consistent with this metric is a complex structure $J_a^{(0)b}=\epsilon_{ac}g^{cb}\sqrt{g}$, and in the harmonic coordinates $z, \bar{z}$, which satisfy the equation $\partial z/\partial\xi^a = iJ_a^b\partial z/\partial\xi^b$, the metric would be of the form $ds^2=\rho dz d\bar{z}$. We now choose a basis $f_i(z)(dz)^2, i=\Gamma,..., 3p-3$ in the space of holomorphic quadratic differentials on S_p and its dual basis

$$\eta^k(z,\bar{z})\frac{d\bar{z}}{dz},\quad k=1,\ldots,3p-3$$

in the space of Beltrami differentials: $\int\eta^k f_j d^2\xi=\delta_j^k$. All the complex structures close to $J^{(0)}$ can then be parametrized by the coordinates $y_i, \bar{y}_i$, so that a complex structure with the coordinates $y_i, \bar{y}_i$ is consistent with the metric $ds^2(y)=\rho|dz+y_i\eta^i d\bar{z}|^2$. We know[3] that $y_i, \bar{y}_i$ are complex-analytic coordinates on M_p. It was shown in Ref. 2 that the measure in the ESVM is

$$Z_p=\int_{M_p} d\mu_p,\quad d\mu_p = F(y)dv\wedge\overline{F(y)dv}(\det\operatorname{Im}\tau)^{-13}$$
$$dv = dy_1\wedge\ldots\wedge dy_{3p-3};\quad p\geqslant 2, \tag{1}$$

where τ is the period matrix (more on this below), and $F(y)dv$ is a holomorphic form on M_p $(3p-3,0)$ which does not vanish anywhere and which has a second-order pole at the infinities D_q, $q=0,1,\ldots,[p/2]$, of the space M_p, where the surface S_p decays into a surface of type q and $p-q$. Here D_0 contains surfaces with a degenerate handle.

2. The Spaces M_2 and M_3. The period matrix τ is defined in the following way. We consider on a surface S_p of type p a symplectic basis of $2p$ cycles (closed, noncontractable paths) $a_i, b_i, i=1,\ldots,p$:

$$a_i\circ a_j=b_i\circ b_j=0,\quad i\neq j;\quad a_i\circ b_j=\delta_{ij}, \tag{2}$$

where $\circ$ represents the algebraic number of intersections of cycles. Associated with the basis $\{a_i,b_i\}$ is the basis $\omega_i=\varphi_i(z)dz, i=1,...,p$ of holomorphic 1-differentials which satisfy the conditions

$$\oint_{a_i}\omega_j=\delta_{ij}. \tag{3}$$

The matrix

$$\tau_{ij}=\oint_{b_i}\omega_j \tag{4}$$

is called the "period" matrix of the surface S_p. We know that we have

$$\tau_{ik} = \tau_{ki}, \quad \operatorname{Im}\tau > 0; \tag{5}$$

i.e., the matrices τ lie in the space H_p of all matrices that satisfy (5). It is not difficult to show that the symplectic basis $\{a_i, b_i\}$ is determined unambiguously by conditions (2), and a given complex structure may correspond to matrices which can be obtained from each other through transformations from the modular group $\Gamma_p = \mathrm{Sp}(p,\mathbb{Z})$ of integer $2p \times 2p$ matrices

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

that satisfy $AB^{\mathrm{T}} - BA^{\mathrm{T}} = CD^{\mathrm{T}} - DC^{\mathrm{T}} = 0, AD^{\mathrm{T}} - BC^{\mathrm{T}} = 1$. Here Γ_p acts on H_p in accordance with

$$M(\tau) = (A\tau + B)(C\tau + D)^{-1}. \tag{6}$$

The complex dimensionality of H_p is $p(p+1)/2$, and for $p = 1, 2, 3$ it is the same as the dimensionality of M_p. In such cases, M_p can be represented by the fundamental region

$$\mathfrak{S}_p = H_p/\Gamma_p$$

of the modular group Γ_p in the upper Siegel half-plane of H_p; i.e.,

$$M_p = \mathfrak{S}_p, \quad p = 1, 2, 3.$$

3. *Measure for $p = 2,3$.* Using the results in Secs. 1 and 2, we can seek the measure in the form

$$d\mu_p = d\nu_p \,|\chi_{12-p}(\tau)|^{-2} (\det \operatorname{Im}\tau)^{p-12}. \tag{7}$$

Here

$$d\nu_p = \prod_{k \leqslant j} \frac{i}{2} d\tau_{kj} \wedge d\overline{\tau}_{kj} (\det \operatorname{Im}\tau)^{-(p+1)} \tag{8}$$

is a modular-invariant measure on H_p. From the condition for the invariance of (7), under modular transformations (6) we find

$$\chi_{12-p}(M(\tau)) = [\det(C\tau + D)]^{12-p} \chi_{12-p}(\tau). \tag{9}$$

For $p = 2$, χ_{10} is therefore a modular form of weight 10 which has no zeros in $\mathfrak{S}_2$. Furthermore, at the infinity D_0 (τ_{11} or $\tau_{22} \to i\infty$) the measure $\prod_{i \leqslant j} d\tau_{ij}$ has a first-order pole. It therefore follows from the analytic properties of the measure,[2] which are given in Sec. 1, that the form χ_{10} has a first-order pole at D_0 and a second-order pole at D_1 ($\tau_{12} \to 0$). In other words, it is parabolic. The form χ_{10} is determined unambiguously by the weight, the order, and the positions of the zeros; it is[4]

$$\chi_{10}(\tau) = \prod_{\mathrm{m}} \theta_{\mathrm{m}}^2(\tau), \tag{10}$$

where the theta constants are defined by

$$\theta_{\mathbf{m}}(\tau) = \sum_{\mathbf{n}\in \mathbb{Z}^p} \exp\left\{\pi i\left(\mathbf{n} + \frac{\mathbf{m}'}{2}\right)^{\mathrm{T}} \tau\left(\mathbf{n} + \frac{\mathbf{m}'}{2}\right) + 2\pi i\left(\mathbf{n} + \frac{\mathbf{m}'}{2}\right)^{\mathrm{T}} \frac{\mathbf{m}''}{2}\right\}$$

$$\mathbf{m} \equiv (\mathbf{m}', \mathbf{m}''), \qquad (11)$$

$p=2$. The components of the vectors $\mathbf{m}'$ and $\mathbf{m}''$ of the characteristic $\mathbf{m}$ take on the values 0,1. The number $e(\mathbf{m}) = (\mathbf{m}')^{\mathrm{T}} \cdot \mathbf{m}'' \,(mod\ 2)$ is called the "parity" of characteristic $\mathbf{m}$, and the product in (10) is over all even characteristics. For type p there are $2^{p-1}(2^p+1)$ even characteristics and $2^{p-1}(2^p-1)$ odd characteristics, and we have $\theta_{\mathbf{m}} \equiv 0$ for $e(\mathbf{m}) = 1$. Using (11), we can easily verify that χ_{10} has a first-order zero at D_0 and a second-order zero at D_1. It can also be shown that χ_{10} does not vanish in $\mathfrak{S}_2$. There is an analogous expression for $p=3$:

$$\chi_9^2(\tau) = \prod_{\mathbf{m}} \theta_{\mathbf{m}}(\tau), \qquad (12)$$

where the product is again over all the even characteristics, of which there are now 36. We do not have room here to present the proof. Equations (7), (10), and (12) constitute the solution of the problem of calculating the measure in the ESVM for type[1)] $p = 2$, 3.

We might also note that in the case of type 2, the space M_2 can be parametrized by the coordinates $(\lambda_1, \lambda_2, \lambda_3)$ of the branch points of the curve

$$y^2 = z(z-1)(z-\lambda_1)(z-\lambda_2)(z-\lambda_3) \qquad (13)$$

in $\mathbb{C}^2 = (y,z)$. In terms of these coordinates, the measure is

$$d\Omega = \prod_{i \leqslant j} d\tau_{ij}(\chi_{10}(\tau))^{-1}$$

$$= d\lambda_1 d\lambda_2 d\lambda_3 [\chi_{10}(\tau)]^{-13/10}((\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)\lambda_1\lambda_2\lambda_3$$

$$\times (1-\lambda_1)(1-\lambda_2)(1-\lambda_3))^{-2/5}, \qquad (14)$$

where τ is determined from λ_1, λ_2, and λ_3 by hyperelliptic integrals. In the partition function

$$Z_3 = \int d\Omega \wedge \overline{d\Omega}\, (\det \operatorname{Im} \tau)^{-13} \qquad (15)$$

we can integrate along each $d^2\lambda_i$ over the entire complex plane, since we are taking each surface into account a *finite* number of times (in general, 720 times).

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1) A less explicit expression, but for arbitrary p, was derived in Ref. 5.

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